A VARIATION ON THE STONE-WEIERSTRASS
THEOREM
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If \( X \) is a set, let \( I^X \) be all functions from \( X \) into the unit interval \( I \). Note that if \( f \) and \( g \) are in \( I^X \) then so are \( 1-f \) and \( fg \). Such a collection of functions is said to have property \( V \). That is, \( F \) has property \( V \) in case

(i) \( F \subseteq I^X \) for some set \( X \),
(ii) \( f \in F \) implies \( 1-f \in F \),
(iii) \( f, g \in F \) implies \( fg \in F \).

Giving \( I^X \) the topology of uniform convergence, we have that the closure of a set with property \( V \) has property \( V \), as does the intersection of such sets. Thus every subset of \( I^X \) is contained in a smallest set with property \( V \), and in a smallest closed set with property \( V \). If \( X \) is a topological space then the set \( D(X) \) of all continuous functions from \( X \) into \( I \) is closed and has property \( V \). The idea of considering such collections of functions comes from a statement of von Neumann in [1]. Essentially, he claims without proof what we give here as a corollary to Theorem 2. I am indebted to Dr. R. S. Pierce for bringing the problem to my attention.

Definition. If \( n \) is a positive integer, let \( P_n \) be the smallest subset of \( D(I^n) \) that has property \( V \) and contains the \( n \) projections.

Lemma 1. Let \( F \) have property \( V \), \( p \in P_n \), and \( f_k \in F \) for \( k = 1, 2, \ldots, n \). Then the function \( f \) defined by

\[
  f(x) = p(f_1(x), f_2(x), \ldots, f_n(x))
\]

is in \( F \).

Proof. Let \( Q \) be the set of all \( q \in D(I^n) \) for which \( q(f_1(x), f_2(x), \ldots, f_n(x)) \) is in \( F \). Then \( Q \) has property \( V \) and contains the \( n \) projections. So \( Q \) contains \( P_n \).

Lemma 2. If \( a < b \) and \( \epsilon > 0 \), then there exists \( p \in P_1 \) such that

- \( p > 1 - \epsilon \) in \([0, a]\),
- \( p < \epsilon \) in \([b, 1]\).

We set \([0, a] = \emptyset \) if \( a < 0 \) and \([b, 1] = \emptyset \) if \( b > 1 \).

Proof. Since for a sufficiently large integer \( k \), \( x^k(1-x)^k < \epsilon \) and

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1 − \(x^k(1−x)^k\) > 1 − \(\varepsilon\) for all \(x\) in \(I\), we can assume that 0 ≤ \(a\) and \(b\) ≤ 1. Also, since there exist \(a', b'\) such that \(a < a' < b' < b\), we can assume that 0 < \(a\) < \(b\) < 1. Now our solution will be of the form \(\rho(x) = (1−x^m)^n\).

Pick \(r\) such that \(\left(\frac{3}{4}\right)^r < \varepsilon\). Pick \(m, s\) such that

\[
\left(\frac{3}{4}\right)^r < s < \frac{1}{b^m} < \left(\frac{\varepsilon}{r}\right)\frac{1}{a^n}.
\]

Let \(n = rs\) and note that \(na^m < \varepsilon, \frac{3}{4} < sb^m < 1.\) So

\[
(1 − a^m)^n > 1 − na^m > 1 − \varepsilon,
\]

\[
(1 − b^m)^n = [(1 − b^m)^s]^r < \left[1 − sb^m + \frac{1}{2} (sb^m)^2\right]^r < \left(\frac{3}{4}\right)^r < \varepsilon.
\]

One can prove by induction that if 0 < \(x\) < 1 then \((1−x)^n < 1−nx + \frac{1}{4}(nx)^2.\)

**Lemma 3.** If \(a_k, b_k \in I\) for \(k = 1, 2, \ldots, n\) then

\[
\prod_{1}^{n} a_k - \prod_{1}^{n} b_k = \sum_{1}^{n} |a_k - b_k|.
\]

**Proof.** The induction step can be verified as follows. Let
\(a = a_1a_2 \cdots a_{n-1}\) and \(b = b_1b_2 \cdots b_{n-1}.\) So \(a, b \in I\) and

\[
|a_{n-k} - bb_n| \leq |a_{n-k} - ba_n| + |ba_n - bb_n|
\]

\[
\leq |a - b| + |a_n - b_n|.
\]

**Lemma 4.** Let \((a, b) \in I \times I\) and \(\varepsilon, \delta > 0.\) Then there exists \(p \in P_3\) such that

\[
\rho(x, y) > 1 − \varepsilon \text{ if } (x − a)^2 + (y − b)^2 \leq \delta^2,
\]

\[
\rho(x, y) < \varepsilon \text{ if } (x − a)^2 + (y − b)^2 \geq (4\delta)^2.
\]

**Proof.** Let the functions \(p_1, p_2, p_3, p_4 \in P_1\) correspond by Lemma 2 to \(a − 2\delta < a − \delta, a + \delta < a + 2\delta, b − 2\delta < b − \delta, b + \delta < b + 2\delta\) and \(\varepsilon/4 > 0,\) respectively. Then let \(p\) be given by

\[
\rho(x, y) = [1 − p_1(x)]p_2(x)[1 − p_4(y)]p_4(y).
\]

**Lemma 5.** Let \(A, B \subseteq I \times I\) be closed and disjoint. If \(\varepsilon > 0\) and \(p \in P_3,\) then there exists \(q \in P_2\) such that

\[
q \geq p \text{ in } I \times I,
\]

\[
q > 1 − \varepsilon \text{ in } A,
\]

\[
q < p + \varepsilon \text{ in } B.
\]
PROOF. We can assume that \( A \) and \( B \) are nonvoid. Let \( 4\delta = \text{dist}(A, B) \). Then \( \delta > 0 \) and there exist \((c_k, d_k) \in A \) for \( k = 1, 2, \ldots, n \) such that the \( \delta \)-neighborhoods of the \((c_k, d_k)\) cover \( A \). For each \( k \) there exists \( q_k \in P_2 \) such that
\[
q_k(x, y) > 1 - \frac{\epsilon}{n} \text{ if } (x - c_k)^2 + (y - d_k)^2 \leq \delta^2,
q_k(x, y) < \frac{\epsilon}{n} \text{ if } (x - c_k)^2 + (y - d_k)^2 \geq (4\delta)^2.
\]
Let \( q_0 = (1 - q_1)(1 - q_2) \cdots (1 - q_n) \). It is clear that \( q_0 > 1 - \epsilon \) in \( B \), and \( q_0 < \epsilon/n \) in \( A \). Now let \( q = 1 - (1 - p)q_0 \). In \( I \times I \) we have \( q \geq 1 - (1 - p) \) in \( A \) and \( q \geq 1 - q_0 > 1 - \epsilon \). And in \( B \) we have \( q - p = 1 - q_0 + pq_0 - p = (1 - q_0)(1 - p) < \epsilon \).

**Theorem 1.** Let \( X \) be a set and \( F \) a closed subset of \( I^X \). If \( F \) has property \( V \) then \( F \) is a lattice.

**Proof.** In view of Lemma 1, it is enough to prove that the functions \((x, y) \rightarrow x \lor y\) and \((x, y) \rightarrow x \land y\) of \( I \times I \) into \( I \) can be uniformly approximated by members of \( P_2 \). Since \( x \lor y = 1 - (1 - x) \land (1 - y) \), it is enough to check \( x \land y \). Let \( 0 < \epsilon < \frac{1}{4} \) and let \( C \) be the set of all \((x, y) \in I \times I \) for which \( \epsilon \leq x \land y \leq 1 - \epsilon \). Then \( C \) is closed and there exists \( m > 0 \) such that \( x^m y^m < \epsilon \) in \( C \). Let \( p(x, y) = 1 - x^m y^m \). Then \( 1 - \epsilon < p < 1 \) in \( C \). For \( k \geq 0 \) let
\[
A_k = \{ (x, y) \in C | p^k(x, y) \geq x \land y \},
B_k = \{ (x, y) \in C | p^k(x, y) \leq x \land y \}.
\]
Then \( A_1 = C \) and for \( k \geq 0 \)
\[
A_k \supset A_{k+1}, \quad B_k \supset C - A_k, \quad A_{k+1} \cap B_k = \emptyset.
\]
Because the \( A_k \) have void intersection, there exists \( n > 2 \) such that \( A_n = \emptyset \). For \( k = 1, 2, \ldots, n \) pick \( q_k \in P_2 \) such that \( q_k \geq p \) in \( I \times I \), \( q_k > 1 - \epsilon/n \) in \( B_{k-1} \), and \( q_k < p + \epsilon/n \) in \( A_k \). Let \( q = q_1 q_2 \cdots q_n \). Now \( C = \bigcup_{i=1}^{n-1} (A_k - A_{k+1}) \). For \( k = 1, 2, \ldots, n - 1 \) we have in \( A_k - A_{k+1} \)
\[
0 \leq p^k - x \land y < p^k - p^{k+1} = p^k(1 - p) < \epsilon.
\]
Also, we have
\[
| p^k - q | \leq | p^k - p^{k+1} | + | p^{k+1} - q | < \epsilon + \left| p^{k+1} - \prod_{i=1}^{n} q_i \right|
\leq \epsilon + \sum_{i=1}^{k} | p - q_i | + | p - q_{k+1} | + \sum_{i=k+2}^{n} | 1 - q_i |
< \epsilon + k \frac{\epsilon}{n} + (1 - p) + (n - k - 1) \frac{\epsilon}{n} < 3\epsilon.
\]
Thus in \( C \), \(|q - x \land y| < 4\varepsilon\). Now by Lemma 5, there exists \( g' \in P_2 \) such that \( g' \geq q \) in \( I \times I \), \( g' > 1 - \varepsilon \) if \( x \land y \geq 1 - \varepsilon \), and \( g' < q + \varepsilon \) if \( x \land y \leq 1 - 2\varepsilon \). Clearly \(|g' - x \land y| < 6\varepsilon\) if \( x \land y \leq \varepsilon \). Similarly, there exists \( g'' \in P_2 \) such that \( g'' \leq q \) in \( I \times I \), \( g'' < \varepsilon \) if \( x \land y \leq \varepsilon \), and \( g'' > q' - \varepsilon \) if \( x \land y \geq 2\varepsilon \). So \(|g'' - x \land y| < 8\varepsilon\) in all of \( I \times I \).

**Theorem 2.** Let \( X \) be a compact space and \( F \) a closed, point-separating subset of \( D(X) \) that has property \( V \). If \( S \) is the set of points of \( X \) taken into the doubleton \( \{0, 1\} \) by every member of \( F \), then \( F \) consists of all functions \( f \in D(X) \) for which \( f(S) \subseteq \{0, 1\} \).

**Proof.** It is well known \([2]\) that for a compact space \( Y \), a closed sublattice of \( C(Y) \) contains any continuous function which it approximates at each pair of points. So, let \( f \in D(X) \) be such that \( f(S) \subseteq \{0, 1\} \), and let \( u, v \) be distinct elements of \( X \). The case when \( X \) has only one element is straightforward.

Suppose \( u, v \in S \). Then there exists \( g \in F \) such that \( g(u) \neq g(v) \), and then one of \( g, 1 - g, 1 - g(1 - g) \) duplicates \( f \) on \( u \) and \( v \).

If \( u, v \in S \), \( u, v \in S \), then there exists \( g \in F \) such that \( g(u) = g(v) \) and \( g(v) \in (0, 1) \). As can be seen from Lemmas 1 and 2, something of the form \( 1 - (1 - g^n) \) will do.

If \( u, v \in S \), then there exist \( g_1, g_2, g_3 \in F \) such that \( g_1(v) \leq g_1(u) \in (0, 1) \), \( g_2(u) \leq g_2(v) \in (0, 1) \), and \( g_3(v) < g_3(u) \). If we let \( h_1 = g_1 g_3 \) and \( h_2 = g_2(1 - g_3) \), we have \( h_1(v) < h_2(u) < 1 \) and \( h_2(u) < h_2(v) < 1 \). Something of the form \( f_2 = (1 - h_2^m) \) approximates \( f \) at \( u \) and \( f \) at \( v \). Something of the form \( f_1 = (1 - h_1^n) \) approximates \( f \) at \( u \) and \( 1 \) at \( v \). So \( f_1 f_2 \) approximates \( f \) at \( u \) and \( v \).

**Corollary.** The smallest closed subset of \( D(I^n) \) having property \( V \) and containing the projections and at least one constant \( c \in (0, 1) \) is \( D(I^n) \) itself.

**References**


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