ON COMMUTATORS IN A SIMPLE LIE ALGEBRA

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1. Introduction. K. Shoda [3] has shown that any \( n \times n \) matrix of trace zero over a field of characteristic zero is expressible as an additive commutator. A. Albert and B. Muckenhoupt [1] have extended this result to fields of all characteristics. Rephrasing their result, it is easily seen that every element of a Lie algebra of type \( A_n \) can be expressed as a Lie product. It seems natural to ask whether a similar assertion is valid for a wider class of Lie algebras.

The purpose of this paper is to employ a generalization of Shoda’s method to show that any element of a classical Lie algebra (defined by R. Steinberg [4]) can be expressed as a Lie product provided only that a certain restrictive assumption on the cardinality of the field \( K \) over which the algebra is defined is satisfied. This restriction is made in this paper for the sole purpose of permitting a relatively simple method to be applied to prove the main theorem.

2. The notation and method of proof. The classical simple Lie algebras will be denoted by \( L = L^*/Z \) where \( Z \) is the center of \( L^* \). \( L^* \) will also be called classical. \( K \) will denote the base field, \( xy \) is the Lie product of \( x \) and \( y \), \( H \) a standard Cartan subalgebra, \( \phi \) runs through the nonzero roots of \( L^* \) relative to \( H \), \( L^* = H + E \), a vector space direct sum, where \( E = \sum_{\phi} L_{\phi} \). Each \( L_{\phi} \) is one-dimensional, and \( H \) is spanned by \( h_i = h_{\phi_i} \) \((i = 1, \ldots, n)\), where \( \phi_i(h_i) = 2, \ h_i = e_{\phi_i}e_{-\phi_i} \), for some \( e_{\phi_i} \in L_{\phi_i}, \ e_{-\phi_i} \in L_{-\phi_i} \), the \( \phi_i \) being a fundamental system of simple roots for \( L^* \) relative to \( H \). If \( \phi \) and \( \psi \) are roots, \( \phi \neq -\psi \), then \( L_{\phi}L_{\psi} = L_{\phi+\psi} \) if \( \phi + \psi \) is a root, and 0 otherwise. \( l \) and \( l' \) will be said to be conjugates of one another if \( l' \) is the image of \( l \) under an automorphism of \( L^* \). The difference \( r - q \), where \( \psi - r \chi \) and \( \psi + q \chi \) are roots of \( L^* \), but \( \psi - (r + 1) \chi \) and \( \psi + (q + 1) \chi \) are not roots, is denoted by \( A_{\chi \psi} \), and \( A_{\phi \psi} \) is abbreviated \( A_{ij} \). The algebras of types \( A_1, B_n, C_n, F_4, \) and \( G_2 \) are not included in the list of classical Lie algebras if \( K \) is of characteristic \( p = 2 \). \( G_2 \) is also excluded if \( p = 3 \). The excluded algebras either are not simple or are isomorphic to other classical

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Presented to the Society, April 14, 1962; received by the editors April 17, 1962 and, in revised form, August 17, 1962.

1 This paper is a portion of the author’s doctoral dissertation. The author would like to express his sincere appreciation to Professor I. N. Herstein for proposing this problem to him and for his helpful suggestions during its preparation. Written with the support of Grant DA-ARO (D)-31-124-G86.

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algebras. Because of these exclusions, if \( \chi \neq \pm \psi, \ A_{x^\ell} \equiv 0 \pmod{p} \) if and only if \( A_{x^\ell} = 0 \).

The aforementioned proof by Shoda was based on two lemmas, namely that any matrix all of whose diagonal elements are zero is expressible as a commutator, and secondly that any matrix of trace zero is similar to a matrix all of whose diagonal elements are zero. Such a matrix is an element of a Lie algebra of type \( A_n \) belonging to the subspace \( E \). We shall use this fact to rephrase Shoda's lemmas in the terminology of Lie algebra theory as follows:

**Lemma I.** There exists an \( h \in H \) such that \( \{ hl | l \in L \} = E \).

**Lemma II.** Every element of \( L \) has a conjugate in \( E \).

For any algebra \( L \) for which Lemma I and Lemma II are valid, we can prove

**Theorem A.** Every element \( s \) of \( L \) can be expressed as a Lie product.

**Proof.** There exists an automorphism \( g \) of \( L \) such that \( g(s) = hl \). Hence \( s = g^{-1}(hl)g^{-1}(l) \).

We shall prove that Lemma I is valid for all classical algebras provided that the cardinality of \( K \) exceeds \( c \) where \( c = 2n - 1 \) for \( A_n \), \( 4n - 5 \) for \( B_n \) and \( C_n \), \( 4n - 7 \) for \( D_n \), \( 5 \) for \( G_2 \), \( 15 \) for \( F_4 \), \( 21 \) for \( E_6 \), \( 33 \) for \( E_7 \), and \( 57 \) for \( E_8 \).

Lemma II will be proved valid for all classical Lie algebras. Theorem A is therefore valid for all classical algebras for which Lemma I is valid.

Let \( L = L_1 \oplus \cdots \oplus L_n \). Suppose that Theorem A is valid for \( L_i \) for all \( i \). Then \( s \in L \) can be written \( s = s_1 + \cdots + s_n = x_1y_1 + \cdots + x_ny_n = (\sum_{i=1}^n x_i)(\sum_{i=1}^n y_i) \), and Theorem A is also valid for \( L \).

3. **Proof of Lemma I.** Consider an element \( a = \sum_\phi m_\phi e_\phi \). It is necessary to find an \( h \in H \) such that \( h(n_\phi e_\phi) = m_\phi e_\phi \) can be solved for \( n_\phi \) for all \( \phi \). Let \( h = \sum_{i=1}^n q_i h_i \). Then \( h(n_\phi e_\phi) = (\sum_i q_i \phi(h_i)) n_\phi e_\phi \). Therefore \( h(n_\phi e_\phi) = m_\phi e_\phi \) can be solved for \( n_\phi \) if \( \sum_i q_i \phi(h_i) \neq 0 \). Let \( \phi(h_i) x_i = g_\phi(x_1, \cdots, x_n) \), a polynomial in \( n \) variables. Since there exists an \( h_i \) such that \( \phi(h_i) \neq 0 \), \( g_\phi \) is not identically zero. Since \( g_\phi = -g_{-\phi} \), we note that if \( \phi(q_1, \cdots, q_n) \neq 0 \) where \( h = \prod_\phi g_\phi \), then \( g_\phi(q_1, \cdots, q_n) \neq 0 \) for all \( \phi \). Such \( q \) exist provided only that the cardinality of \( K \) exceeds the degree of \( h(x_1, \cdots, x_n) \) in \( x_i \) for all \( i \). This degree can be calculated and found to be \( c \), the number defined above. The validity of Lemma I for \( L^* \) implies its validity for \( L \). Therefore, with the exceptions stated above, Lemma I is valid for all classical Lie algebras.
4. Proof of Lemma II.

**Lemma 4.1.** If \( a = \sum_{i=1}^{n} m_i h_i + \sum_{\phi} m_{\phi} e_{\phi} \in \mathbb{Z} \), then \( a \) has a conjugate \( a' = \sum_{i=1}^{n} m_i' h_i + \sum_{\phi} m_{\phi}' e_{\phi} \) such that \( m_j' = 0 \) for some \( j \).

**Proof.** We let \( x_\phi(y) \) be the automorphism of \( L^* \) having the same effect as \( \exp(\text{ad } y e_\phi) \) on each generator, except that \( x_\phi(y)e_{-\phi} = e_{-\phi} + y h_\phi + y^2 e_\phi \) if \( K \) has characteristic 2.

Suppose that there exists a root \( \psi \) such that \( m_\psi \neq 0 \). Let \( a' = \sum_{i=1}^{n} m_i' h_i + \sum_{\phi} m_{\phi}' e_{\phi} = x_{-\psi}(y)a \). Then

\[
m_j' = m_j + k_j \frac{A_{\psi \phi}}{A_{\phi \psi}} m_{\psi} y,
\]

where \( \psi = \sum_{i=1}^{n} k_i \phi_i \). It is easily observed that \( k_j(A_{\psi \phi}/A_{\phi \psi}) = 0 \) (mod \( p \)) does not hold for every \( j \). Therefore there is a \( j \) such that \( m_j' = 0 \) can be solved for \( y \), and consequently Lemma 4.1 is valid if \( a \notin H \).

Now suppose \( m_\phi = 0 \) for all \( \phi \neq 0 \), i.e., \( a \in H \). If it can be shown that \( a \) is conjugate to an element \( a' \in H \), then the lemma will follow by the transitivity of the conjugacy relation. Since \( x_\phi(1)a = a' = \sum_{i=1}^{n} m_i' h_i + \sum_{\phi} m_{\phi}' e_{\phi} \) where \( m_{\phi}' e_{\phi} = \sum_{i=1}^{n} m_i h_i e_{\phi} = a e_{\phi}, m'_i = 0 \) for all \( \phi \) only if \( a e_\phi = 0 \) for all \( \phi \). Since \( a \in H \) and \( H \) is abelian, \( a h = 0 \) for \( h \in H \). Thus the Lie product of \( a \) with any generator would be zero, implying \( a \in Z \), thus contradicting the hypothesis of the lemma.

**Lemma 4.2.** Let \( S \) be an indecomposable subset of the fundamental system of roots for a classical Lie algebra \( L^* \). Then the subalgebra \( L_S^* \) generated by \( \{e_\phi, e_{-\phi}\} \) for \( \phi \in S \) is a classical Lie algebra unless \( K \) is of characteristic 2, and \( S \) consists of only one root.

**Proof.** Let \( L \) be a complex simple Lie algebra with Cartan subalgebra \( H \) and fundamental system of roots \( \phi_1, \ldots, \phi_n \). Let \( S \) be an indecomposable subset of this fundamental system. Let \( L_S \) be the subalgebra of \( L \) generated by \( \{e_\phi, e_{-\phi}\} \) for \( \phi \in S \). Let \( R \) be the linear transformation from the complex vector space \( V_S \) spanned by \( \phi \in S \) into the space of linear functionals on \( H_S = H \cap L_S \), the space spanned by \( \{h_i\} \) where \( \phi_i \in S \), defined by \( R(\lambda) = \lambda | H_S \). If \( \lambda = \sum c_i \phi_i \) is in the kernel of \( R \), then \( \sum c_i \phi_i(h_i) = 0 \) for all \( j \) such that \( \phi_j \in S \). Since the matrix \( (\phi_i(h_j)) = (A_{ij}) \) for \( i, j \) such that \( \phi_i, \phi_j \in S \) is a principal submatrix of the Cartan matrix of \( L \), it is nonsingular by [2], and so \( \lambda \) must be zero, and \( R \) is one-to-one. Thus if the root \( \psi \) is in \( V_S \), there is an \( h \in H_S \) such that \( h e_\psi = \psi(h) e_\phi \neq 0 \), and so \( H_S = \{l \in L_S : l h = 0 \text{ for all } h \in H_S\} \), i.e., \( H_S \) is a Cartan subalgebra of \( L_S \). Clearly, \( \psi | H_S \) is a root of \( H_S \) in \( L_S \), and, conversely, every root of \( H_S \) in \( L_S \)
is the restriction to $H_S$ of a root in $V_S$ since the elements $e_\varphi, \varphi$ a root in $V_S$, form a basis for $E_S = E \cap L_S$, and $L_S = H_S + E_S$. Since $R(\varphi - \chi)$ is a root of $H_S$ in $L_S$ if and only if $\varphi - \chi \in V_S$ is a root of $H$ in $L$, and since $R$ preserves linear independence, \( \{\varphi_i | H_S : \varphi_i \in S\} \) is a fundamental system of simple roots for $L_S$. Therefore since $S$ is indecomposable, $L_S$ is simple.

Let $L_Z$ be the Lie subring of $L$ consisting of all integral linear combinations of the basis \( \{e^\varphi, h_i\} \). Then $\tilde{L} = L_Z \otimes_{\mathbb{Z}} K$ is a Lie algebra over the base field $K$ if $(l_1 \otimes k_1)(l_2 \otimes k_2) = (l_1 l_2 \otimes k_1 k_2)$. A comparison of their multiplication tables, identifying $l \otimes k$ with $kl$, reveals that $\tilde{L}$ is isomorphic to the classical algebra $L^*$ defined in §2 except when the type of $L$ and the characteristic of $K$ are specifically excluded by that definition. Similarly, $\tilde{L}_S = (L_S)_Z \otimes_{\mathbb{Z}} K$ is classical, with the same exceptions, since $L_S$ is simple. However, it is easily observed that the only algebra among these nonclassical algebras which can be a subalgebra of a classical algebra is the algebra of type $A_1$ over a field of characteristic 2. This establishes the lemma.

Similarly, if $S$ is decomposable, $L_S^*$ is a direct sum of classical Lie algebras unless $K$ has characteristic 2, and $S$ has a maximal indecomposable subset containing only one root.

In order to establish Lemma II, we first prove

**Lemma II'.** Every noncentral element of $L^*$ has a conjugate in $E$.

The proof of Lemma II' will proceed by induction on the number of simple roots. First we observe that if $K$ has characteristic $p > 2$, then Lemma II' is valid for $A_1$ by Lemma 4.1.

If $p = 2$, we establish Lemma II' for $A_2$ as follows: Let $i = 1, j = 2$ or $i = 2, j = 1$. Suppose $a = \sum m_i h_i + \sum m_\varphi e_\varphi$. Let $m_i \neq 0$. We may assume $m_{\varphi_i} \neq 0$ or $m_{-\varphi_i} \neq 0$ since if $m_{\varphi_i} = m_{-\varphi_i} = 0$, $x_{\varphi_i}(1)a = a' = \sum m_i' h_i + \sum m_{\varphi_i} e_\varphi$ where $m_{\varphi_i}' \neq 0$ and $m_i' = m_i$. An appropriate value of $y$ can be found for either $x_{\varphi_i}(y)a$ or $x_{-\varphi_i}(y)a$ to yield an element with $m_i' = m_i$. If $m_{\varphi_1} + m_2 = m_{-\varphi_1} - m_2 = 0$, $x_{\varphi_1} + \varphi_2(1)$ will yield $a'$ with $m_{\varphi_1} + m_{\varphi_2} \neq 0$ and $m'_i = m_i$. Therefore suppose $a = \sum m_i h_i + \sum m_{\varphi_i} e_\varphi$, $m_i = m_i$, and $m_{\varphi_1} + m_2 \neq 0$ or $m_{-\varphi_1} - m_2 \neq 0$. Then for appropriate $y \in K$, $x_{\varphi_1} + \varphi_2(y)$ or $x_{-\varphi_1} - \varphi_2(y)$ applied to $a$ yields an element such that $m_i' = m_i = 0$. Hence Lemma II' is valid for $A_2$ over a field $K$ of characteristic 2.

Assume $a \in L^*$, $a \in \mathbb{Z}$, and that the cardinality of the fundamental system of roots of $L^*$ is $n$. If $K$ has characteristic $p > 2$, assume that Lemma II' has been proved for all algebras $L^*$ with fundamental systems of cardinality $c < n$ where $L^*$ is classical or the direct sum of classical algebras. If $p = 2$, assume that Lemma II' has been proved for all algebras with fundamental systems of cardinality $c$ where
Let \( a' = \sum m_i h_i + \sum m_i e_i \) be a conjugate of \( a \) such that the number of \( i \) such that \( m_i \neq 0 \) is minimal. We wish to show \( a' \in E \).

Let \( S \) be the subset consisting of those roots \( \phi_i \) of \( L^* \) for which \( m_j \neq 0 \). It is possible to choose \( a' \) in such a way that \( \sum m_i h_i \) is not in the center \( Z_S \) of \( L_{\phi}^* \). To show this, write \( L_{\phi}^* = L_{\phi_1}^* \oplus \cdots \oplus L_{\phi_r}^* \). Suppose \( \sum m_i h_i \in Z_S \). Let \( \phi_i \in S \), but \( A_{\phi_i} \neq 0 \) for some \( \phi_i \in S_i \), where \( S_i \) consists of the roots \( \phi_i, \ldots, \phi_i \). Since \( A_1 \) has no center if \( p > 2 \), and \( A_1 \) is excluded from the list of classical Lie algebras if \( p = 2 \), we have \( r > 1 \).

Unless \( m_{\phi_1 + \phi_1} = m_{-\phi_1 - \phi_1} = 0 \), automorphisms \( x_{\phi_1 + \phi_1}(y) \) or \( x_{-\phi_1 - \phi_1}(y) \) for appropriate \( y \) map \( a' \) into an element with \( m'_i = 0 \), \( m'_i \neq 0 \), \( m'_i = m_i \) for \( i \neq k, l \). If \( m_{\phi_1 + \phi_1} = m_{-\phi_1 - \phi_1} = 0 \), the automorphism \( x_{\phi_1 + \phi_1} \) maps \( a' \) into an element for which \( m_{\phi_1 + \phi_1} \neq 0 \), thus satisfying the hypothesis of the preceding statement. \( \sum m_i h_i \) summed over \( i \) such that \( \phi_i \in S \) is not in \( Z_S \).

If \( S_i \) is a single root \( \phi_i \) for all \( i \), and \( p = 2 \), then \( L_{\phi_1 + \phi_1} \) where \( A_{\phi_1} \neq 0 \) is \( A_2 \), and it is shown above that any element in \( A_2 \) has a conjugate in \( E \). Since \( x_{\phi_1}(y) \) for \( \phi \in A_2 \) leaves \( m'_i = m_i \) where \( \phi_i \in A_2 \), we have obtained a contradiction of the minimality assumption.

For all other situations we proceed as follows. Let \( a' \in L_{\phi}^* \). Then by induction on the cardinality of the fundamental system, \( a' \) is conjugate to an element in \( E_S = E \). Hence \( a' \in E \).

If \( a' \in L_{\phi}^* \), then \( a' \in L_{\phi}^* + B \) where \( B = \sum L_\phi \), summed over \( \phi \) such that \( L_\phi \subseteq L_{\phi}^* \). If \( L_\phi \subseteq L_{\phi}^* \) and \( L_\phi \subseteq L_{\phi}^* \), then \( x_{\phi}(y)e_\phi \subseteq B \). Therefore \( x_{\phi}(y)b \in B \) for all \( b \in B \).

Now we have \( a' = a'' + b \), \( a'' \in L_{\phi}^* \). By a sequence of automorphisms \( x_{\phi}(y) \) where \( L_\phi \subseteq L_{\phi}^* \), \( a' \) can be transformed into an element in \( E_S + B = E \). This completes the proof of Lemma II'.

Lemma II follows immediately from Lemma II', since any automorphism of \( L^* \) induces an automorphism of \( L \), and an element in the center of \( L^* \) has the image zero in \( L \).

References


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