AN INTEGRAL INEQUALITY
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1. Introduction. The purpose of this note is to derive some integral inequalities. In particular, we give conditions on real-valued integrable functions \( h, g \) and \( \phi \) defined for all \( x \in A \) which imply that

\[
\int_A g \phi \, dx \geq \int_A h \phi \, dx
\]

or equivalently

\[
\int_A (g - h) \phi \, dx = \int_A f \phi \, dx \geq 0
\]

where we set \( f = g - h \). We show that these results are a generalization of an inequality due to P. R. Beesack [1] except for certain integrability restrictions which he does not require. We use our results to obtain a comparison theorem for the lowest eigenvalue of a membrane. The method used in deriving the inequality (1) also yields a generalization of certain mean value theorems for integrals.

All of our results are obtained by use of the following

**Lemma.** Let \( f \) and \( \phi \) be real-valued functions defined for \( x \in A \) with \( f \) integrable over \( A \). Let \( \phi \) be measurable over \( A \) and satisfy the condition

\[-\infty < m \leq \phi(x) \leq M < \infty.\]

Define the sets

\[
A(y) = \{ x : \phi(x) \geq y \}
\]

and

\[
B(y) = A - A(y) = \{ x : \phi(x) < y \}.
\]

Then

\[
\int_A f \phi \, dx = m \int_A f \, dx + \int_m^M \left( \int_{A(y)} f \, dx \right) \, dy
\]

and

\[
\int_A f \phi \, dx = M \int_A f \, dx - \int_m^M \left( \int_{B(y)} f \, dx \right) \, dy.
\]

**Proof.** Define the function \( F \) with values

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\[ F(y) = \begin{cases} \int_{A(y)} f \, dx, & y \in [m, M); \\ 0, & y = M. \end{cases} \]

It follows that

(4) \[ \int_A f \phi \, dx = - \int_m^M y dF(y) \]

since for any partition \( P_n = \{m = y_0 < y_1 < \cdots < y_n = M\} \) with norm \( P_n = \delta < \epsilon / \int_A |f| \, dx \) and \( y_k \in [y_{k-1}, y_k] \) we have

\[ \left| \int_A f \phi \, dx - \sum_{k=1}^n y_k [F(y_{k-1}) - F(y_k)] \right| \leq \delta \int_A |f| \, dx < \epsilon. \]

Integrating the right side of (4) by parts, we get

\[ \int_A f \phi \, dx = -yF(y) \bigg|_m^M + \int_m^M y F(y) \, dy = m \int_A f \, dx + \int_m^M \left( \int_{A(y)} f \, dx \right) \, dy, \]

and (2) is proved. (3) follows immediately from

\[ (M - m) \int_A f \, dx = \int_m^M \left( \int_{A(y)} f \, dx + \int_{B(y)} f \, dx \right) \, dy \]

if we replace \( \int_m^M (\int_{A(y)} f \, dx) \, dy \) by its equivalent from (2).

2. **Inequalities.** In the following, we assume that \( f, \phi \) and \( f \cdot \phi \) have finite integrals over the set \( A \). Our lemma then implies the following results.

**Theorem 1.** Let \(-\infty < m \leq \phi(x) \) for all \( x \in A \) and let \( \int_{A(y)} f \, dx \geq 0 \) for all \( y \in [m, \infty) \). Then the condition \( m \int_A f \, dx \geq 0 \) implies that

\[ \int_A f \phi \, dx \geq 0. \]

**Proof.** Let

\[ \phi_M(x) = \begin{cases} M, & x \in A(M); \\ \phi(x), & x \in A - A(M). \end{cases} \]

It then follows from our hypothesis and (2) that

(5) \[ \int_A f \phi_M \, dx \geq 0. \]
By the Lebesgue dominated convergence theorem

$$\lim_{M \to \infty} \int_A f_M \, dx = \int_A f \, dx.$$  

Hence (5) implies the desired result.

By the same reasoning and (3) we may prove

**Theorem 2.** Let $\phi(x) \leq M < \infty$ for all $x \in A$ and let $\int_{B(y)} f \, dx \leq 0$ for all $y \in (-\infty, M]$. Then the condition $M\int_A f \, dx \geq 0$ implies that

$$\int_A f \, dx \geq 0.$$  

We may combine the results of Theorems 1 and 2 to get

**Theorem 3.** Let $A_1 = \{x : \phi \geq 0\}$ and $A_2 = A - A_1$. If $\int_{A(y)} f \, dx \geq 0$ for all $y \in [0, \infty)$ and $\int_{B(y)} f \, dx \leq 0$ for all $y \in (-\infty, 0)$ then

$$\int_A f \, dx \geq 0.$$  

**Proof.** By Theorem 1, $\int_{A_1} f \phi \, dx \geq 0$. By Theorem 2, $\int_{A_2} f \phi \, dx \geq 0$. Adding these inequalities we get the desired result.

3. **Remarks.** Note that the conditions on $f$ are given only in terms of the sets $A(y)$ and $B(y)$. These sets may be known even though the function $\phi$ is not. This is the case if $\phi$ is symmetric with respect to a point and has either a positive or negative gradient in $A$.

As a special case of our results, we have the theorem due to Beesack [1] when $\phi \cdot (F - G)$ is integrable.

**Theorem (Beesack).** Let $F$, $G$ and $\phi$ be integrable over $A$ and let $E_1 = \{x : F(x) \leq G(x)\}$ and $E_2 = \{x : F(x) > G(x)\}$ and suppose

(6) \hspace{1cm} \int_A G \, dx \leq \int_A F \, dx.

Then if either

(7) \hspace{1cm} 0 \leq \phi(x_1) \leq \phi(x_2)

or

(8) \hspace{1cm} \phi(x_1) \leq 0 \leq \phi(x_2)

for every pair $x_1$, $x_2$ such that $x_1 \in E_1$ and $x_2 \in E_2$

$$\int_A \phi[F - G] \, dx \geq 0.$$  

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We show that the hypothesis of this theorem is a special case of that of Theorems 1 and 3. Let \( f = F - G \) and let \( \bar{y} = \sup_{x \in E_1} \phi(x) \). Then \( \phi(x) \geq \bar{y} \) for all \( x \in E_2 \). Hence \( A(y) \subseteq E_2 \) and therefore

\[
\int_{A(y)} f \, dx = \int_{A(y)} (F - G) \, dx \geq 0
\]

for all \( y > \bar{y} \). For \( y \leq \bar{y} \), we have

\[
\int_{A(y)} (F - G) \, dx = \int_{E_1} (F - G) \, dx + \int_{E_1 \cap A(y)} (F - G) \, dx.
\]

The first integral on the right is positive while the second is negative. If their sum is negative for some value \( y = y_1 \), then it is negative for all \( y \leq y_1 \). But this contradicts condition (6) since \( A(m) = A \). If (7) is satisfied then this implies that the hypothesis of Theorem 1 is also true.

If (8) is true then we have a special case of Theorem 3 since \( F - G \leq 0 \) in \( E_1 \) implies \( \int_{B(y)} (F - G) \, dx \leq 0 \) for \( y < 0 \) and \( F - G \geq 0 \) in \( E_2 \) implies \( \int_{A(y)} (F - G) \, dx \geq 0 \) for \( y > 0 \).

4. A comparison theorem. The following result is typical of a kind that might be derived from our inequality.

**Theorem 4.** Let \( p(x, y) \) and \( q(x, y) \) be non-negative real continuous functions defined in a simply connected domain \( D \) with a piecewise smooth boundary \( C \) such that

\[
\int \int_D p(x, y) \, dx \, dy = \int \int_D q(x, y) \, dx \, dy.
\]

Consider the eigenvalue problems associated with the nonhomogeneous vibrating membrane over \( D \),

\[
\begin{align*}
\nabla^2 u + \lambda p(x, y) u &= 0, \quad u = 0 \text{ on } C, \\
\nabla^2 v + \mu q(x, y) v &= 0, \quad v = 0 \text{ on } C.
\end{align*}
\]

Let \( v_1(x, y) \) denote the eigenfunction corresponding to the lowest eigenvalue \( \mu_1 \) of (10) and define

\[
A(z) = \{(x, y) : [v_1(x, y)]^2 \geq z\}.
\]

If \( \int \int_{A(z)} (p - q) \, dx \, dy \geq 0 \), for all \( z \geq 0 \), then

\[
\lambda_1 \leq \mu_1
\]

where \( \lambda_1 \) is the lowest eigenvalue of (9).
Proof. Since we may choose $v_1$ so that the condition $0 \leq v_1 \leq 1$ is satisfied, Theorem 1 and the above conditions imply

$$\int \int_D p v_1^2 dx dy \geq \int \int_D q v_1^2 dx dy.$$  

Thus we have

$$\mu_1 = \frac{\int \int_D (v_{1x} + v_{1y})^2 dx dy}{\int \int_D v_1^2 dx dy} \geq \frac{\int \int_D (v_{1x} + v_{1y}) dx dy}{\int \int_D p v_1 dx dy} \geq \lambda_1.$$

In terms of a nonhomogeneous vibrating membrane, our theorem says that if the cumulative mass of a membrane with respect to the sets $A(x)$ is greater than that of the other then the first has a lower fundamental tone. We also note that corresponding results hold for problems of different dimensions and for other boundary conditions.

5. Mean value theorems. The mean value theorems stated below are a consequence of our lemma. In the following, we assume that the hypothesis of the lemma is satisfied.

**Theorem 5.** If $0 \leq \int_A \varphi dx \leq \int_A f dx$ for all $y \in [m, M]$ then there exists a number $y \in [m, M]$ such that

$$y \int_A f dx = \int_A f \varphi dx.$$

**Proof.** Since $\int_A \varphi dx \leq \int_A f dx$ implies $\int_A \varphi dx = \int_B \varphi dx \geq 0$, (2) and (3) give the inequalities

$$m \int_A f dx \leq \int_A f \varphi dx \leq M \int_A f dx.$$  

If $\int_A f dx = 0$, $\int_A f \varphi dx = 0$ and the theorem is trivially true; if $\int_A f dx > 0$, then $\gamma = \int_A f \varphi dx / \int_A f dx$.

**Theorem 6.** If $F(y) = \int_A f dx$ is continuous then there is an $\eta \in [m, M]$ such that

$$\int_A f \varphi dx = M \int_{A(\eta)} f dx + m \int_{B(\eta)} f dx.$$

**Proof.** Applying the one dimensional mean value theorem to the last integral of (3) we get
\[
\int_{A} f \phi \, dx = M \int_{A} f \, dx - \int_{B(\varphi)} f \, dx (M - m) \\
= M \int_{A(\varphi)} f \, dx + m \int_{B(\varphi)} f \, dx.
\]

We remark that the hypothesis of Theorem 5 replaces the condition \( f \geq 0 \) in the classical first mean value theorem for the Lebesgue integral (p. 26 of [2]). Theorem 6 is a generalization of the second mean value theorem for the Lebesgue integral since we do not require a monotonicity condition on \( \phi \) (p. 104 of [2]).

**References**


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