Σ-SYMMETRIC LOCALLY CONVEX SPACES

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In [1] it is shown that barrelledness and quasi-barrelledness are merely the two extreme examples of a property, called Σ-symmetry, which may be possessed by a locally convex Hausdorff topological vector space. The object of this note is to show how recent characterisations [2; 3] of barrelled and quasi-barrelled spaces may be subsumed under characterisations of Σ-symmetric spaces, and to exhibit some properties of these spaces. First we need some definitions and simple results.

1. Let $E$ be a locally convex Hausdorff topological vector space (abbreviated to LCS in what follows), and let $Σ$ be a class of bounded subsets of $E$ whose union is $E$. Let $E'$ denote the topological dual of $E$, and let $E'_Σ$ be the set $E'$ endowed with the topology of uniform convergence on the members of $Σ$.

Definition 1. A subset of $E$ is said to be $Σ$-bornivorous if it absorbs every member of $Σ$.

Definition 2. We say that $E$ is Σ-symmetric if any of the following equivalent conditions hold:

(a) Every $Σ$-bornivorous barrel in $E$ is a neighbourhood of zero.
(b) Every bounded subset of $E'_Σ$ is equicontinuous.
(c) The topology induced on $E$ by the strong dual of $E'_Σ$ is the original topology of $E$.

The equivalence of these conditions was proved in [1]. If $Σ_1 ⊂ Σ_2$ it is easy to see that $Σ_1$-symmetry implies $Σ_2$-symmetry; the strongest restriction on $E$ is obtained by taking for $Σ$ the class $s$ of all subsets of $E$ consisting of a single point, and then $Σ$-symmetry is simply the property of being barrelled. If $Σ$ is the class $b$ of all bounded subsets of $E$ we have the weakest $Σ$-symmetric property, which is that of being quasi-barrelled. Whatever the choice of $Σ$, the topology of a $Σ$-symmetric space is the Mackey topology [1].

Definition 3. $E$ is said to be $Σ$-bornological if every convex $Σ$-bornivorous subset of $E$ is a neighbourhood of zero.

It follows easily that $E$ is $Σ$-bornological if and only if every linear map of $E$ into an LCS $F$ which takes members of $Σ$ into bounded sets in $F$ is continuous. For all choices of $Σ$, a $Σ$-bornological space is bornological.

Given any LCS $E$ and any class $Σ$ of bounded sets whose union is

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697
We may define on $E$ a new topology $\mathcal{S}$ by taking as a fundamental system of neighbourhoods of zero the class of all convex circled $\Sigma$-bornivorous subsets of $E$. $\mathcal{S}$ is the finest locally convex topology for which the members of $\Sigma$ remain bounded, and the space $F$ obtained by endowing the point set $E$ with $\mathcal{S}$ is $\Sigma$-bornological. $F$ is called the $\Sigma$-bornological space associated with $E$. We note that when $\Sigma = s$, $\mathcal{S}$ becomes the finest locally convex topology on $E$, and when $\Sigma = b$, $\mathcal{S}$ is the associated bornological topology [4, Chapitre 3, §2, Exercice 13].

**Definition 4.** Let $E_1$, $E_2$ be LCS. A linear mapping $u$ from $E_1$ onto $E_2$ is said to be **almost open** if for every neighbourhood $U$ of zero in $E_1$, the closure of $u(U)$ in $E_2$ is a neighbourhood of zero for the Mackey topology on $E_2$.

**Definition 5.** A linear subspace $Q$ of the dual $E'$ of an LCS $E$ is said to be **almost closed** if for every neighbourhood $U$ of zero in $E$, $U^\circ \cap Q$ is weakly closed in $E'$, where $U^\circ$ denotes the polar of $U$.

2. The theorems in this section give characterisations of $\Sigma$-symmetric spaces. The first result includes Theorems 2.3 and 2.4 of [2] as special cases.

**Theorem 1.** Let $E$ be an LCS and $\Sigma$ a class of bounded subsets of $E$ whose union is $E$. Let $F$ be the $\Sigma$-bornological space associated with $E$. Then $E$ is $\Sigma$-symmetric if and only if the topology of $E$ is the Mackey topology and either of the following conditions holds:

(a) The identity map from $F$ onto $E$ is almost open.

(b) $E'$ is almost closed in $F'$.

**Proof.** This follows the pattern of the corresponding proof in [2]; we give some details for convenience. To prove that $\Sigma$-symmetry implies (b), all we need show, since $F$ has the Mackey topology, is that $E' \cap K$ is weakly closed in $F'$ for every weakly compact convex circled subset $K$ of $F'$. Since $K$ is an equicontinuous subset of $F'$ it is bounded in $F'_K$, so that $E' \cap K$ is bounded in $E'_K$, $E'_K$ being plainly a topological subspace of $F'_K$. Since $E$ is $\Sigma$-symmetric, $E' \cap K$ is thus an equicontinuous subset of $E'$, and is hence relatively weakly compact in $E'$. Actually $E' \cap K$ is weakly compact in $E'$, since $E'_K$ is a topological subspace of $F'_K$, and $K$ is weakly closed in $F'$. It follows that $K \cap E'$ is weakly closed in $F'$.

Condition (a) and the Mackey condition imply $\Sigma$-symmetry, since if $U$ is a $\Sigma$-bornivorous barrel in $E$ it is the closure in $E$ of a neighbourhood of zero in $F$, so that by (a), $U$ is a neighbourhood of zero in $E$.

This completes the proof of the theorem, since Pták [5] has shown that (a) and (b) are equivalent.
The next theorem is a characterisation in terms of the closed graph theorem, and contains Theorems 2.2 and 3.1 of [3] as particular cases.

**Theorem 2.** Let \( E, \Sigma \) be as in Theorem 1. Then \( E \) is \( \Sigma \)-symmetric if and only if for every Banach space \( F \) the following is true:

If \( u \) is any linear map from \( E \) into \( F \) such that

(a) \( u \) takes members of \( \Sigma \) into bounded sets in \( F \);
(b) the graph of \( u \) is closed;
then \( u \) is continuous.

The proof is an obvious modification of that given in [3].

3. We conclude by indicating various general properties of \( \Sigma \)-symmetric and \( \Sigma \)-bornological spaces.

**Theorem 3.**

(a) If \( E \) is \( \Sigma \)-symmetric, \( E' \) is quasi-complete.
(b) Let \( \Sigma \) be a class of convex, circled, closed bounded sets whose union is \( E \). Then if \( E \) is \( \Sigma \)-bornological, \( E' \) is complete.

**Proof.**

(a) Let \( B \) be a bounded closed subset of \( E' \). Since \( E \) is \( \Sigma \)-symmetric, \( B \) is equicontinuous and is therefore complete [4, Chapitre 3, §3, Théorème 4].

(b) The completion of \( E' \) is the set of all linear functionals on \( E \) whose restriction to each member of \( \Sigma \) is continuous [4, Chapitre 4, §3, Exercice 3]. Such functionals are bounded on the members of \( \Sigma \), and since \( E \) is \( \Sigma \)-bornological they are continuous on \( E \). Hence \( E' \) coincides with its completion.

Specialisations of this theorem give, for example, the familiar results that the dual of a quasi-barrelled space is strongly quasi-complete, and that the strong dual of a bornological space is complete.

**Theorem 4.** Let \((E_i)_{i \in I}\) be any family of LCS, and for each \( i \in I \) let \( \Sigma_i \) be a class of convex circled bounded subsets of \( E_i \) whose union is \( E_i \). Let \( \Sigma \) be the class of all subsets of \( E = \prod_{i \in I} E_i \) of the form \( \prod_{i \in I} S_i, S_i \subseteq \Sigma_i \). Then if for all \( i \in I, E_i \) is \( \Sigma_i \)-symmetric, \( E \) is \( \Sigma \)-symmetric when endowed with the usual product topology.

**Proof.** Let \( B \) be any bounded subset of \( E' \). We need to prove that \( B \) is equicontinuous. The topology of \( E' \) is simply the topological direct sum of the topologies of the \((E_i)'_{i \in H} \) [6, Chapitre 4, §1, Proposition 7]. It follows [6, Chapitre 4, §1, Proposition 5] that \( B \) is contained and bounded in the direct sum of a finite number of the \((E_i)'_{i \in H} \), i.e. \( B \subseteq \sum_{i \in H} (E_i)'_{i \in H} \). Hence for each \( i \in H \) the projection of \( B \) into \( (E_i)'_{i \in H} \) is bounded and so is equicontinuous, since \( E_i \) is \( \Sigma_i \)-symmetric. \( B \) is thus an equicontinuous subset of \( E' \) [6, Chapitre 2, §15, Proposition 22, Corollaire 1], which completes the proof.
By choosing the $\Sigma$, suitably we can obtain as special cases of this theorem the well-known results that the product of any family of barrelled (resp. quasi-barrelled) spaces is barrelled (resp. quasi-barrelled.)

**Theorem 5.** Let $E$ be $\Sigma$-symmetric and let $M$ be a closed subspace of $E$. Denote by $\phi$ the canonical mapping $E \to E/M$, and put $\Sigma_1 = \{\phi(S) : S \in \Sigma\}$. Then $E/M$ is $\Sigma_1$-symmetric.

The proof is obvious.

**References**