A NOTE ON ABSOLUTE $G_t$-SPACES

M. REICHAW-REICHBACH

A set $X$, which is a $G_3$ in a compact space\(^1\) is called an absolute $G_t$-space (simply-absolute $G_3$) or a topologically complete space. It was noted by Knaster\(^2\) that there exist two classes $\mathfrak{A}$ and $\mathfrak{B}$ of such spaces, where by definition:

$X \in \mathfrak{A}$ if $X$ is an absolute $G_3$ and there exists a homeomorphism $h: X \to Y$ of $X$ into a compact space $Y$, such that the image $h(X)$ of $X$ can be written in the form:

$\bigcap_{i=1}^{\infty} G_i$, with $\dim Fr(G_i) < \dim X$ and $G_i$ open in $Y$,

$X \in \mathfrak{B}$ if $X$ is an absolute $G_3$ and $X \in \mathfrak{A}$. Knaster also showed\(^4\) i.a., that the set $N \times D$, where $N$ is the set of irrational numbers of the interval $D = [0, 1]$, belongs to $\mathfrak{B}$. In answer to one of his questions, it was proved by Lelek\(^5\) that every set of the form $N \times Z$, where $Z$ is an arbitrary finite-dimensional compact set, belongs to $\mathfrak{B}$. Lelek posed also the following:

**Problem.**\(^6\) Does there exist for every metric, separable and topologically complete, finite-dimensional space $X \in \mathfrak{B}$, with $\dim X > 0$, a compact space $Z$, with $\dim Z > 0$ such that the set $N \times Z$ has a topological image in $X$?

The aim of this paper is to give a negative answer to this problem.

This will be done by the following:

**Example** of a set $X \in \mathfrak{B}$, with $\dim X = 1$, which does not contain a topological image of any set of the form $N \times Z$ with $Z$ compact and $\dim Z > 0$.

Let namely $D_n$ be the closed unit interval joining the points $p_n = (1/n, 0)$ and $q_n = (1/n, 1)$ in the $(x, y)$-plane, $n = 1, 2, \cdots$ (i.e., $D_n = \{ (x, y); x = 1/n, 0 \leq y \leq 1 \}$) and let $p = (0, 0)$ and $q = (0, 1)$.

We put $X = \bigcup_{n=1}^{\infty} D_n \cup (p) \cup (q)$. Evidently $\dim X = 1$, and it suffices to show that the set $X$ has the following properties:

---

\(^1\) Only metric, separable spaces are considered.
\(^2\) See [1, p. 264].
\(^3\) $\dim X$ denotes the dimension of $X$; $Fr(X)$ is the boundary of $X$.
\(^4\) See [1, pp. 263-264].
\(^5\) See [3, p. 34].
\(^6\) See [3, p. 34]. The author learned recently that this problem has also been solved, in an entirely different way, by A. Lelek (unpublished to date).
A NOTE ON ABSOLUTE $G_\delta$-SPACES

(a) $X$ is an absolute $G_\delta$,\footnote{It is easily seen that $X$ is also an absolute $F_{\sigma}$, i.e. an $F_{\sigma}$ in a compact space.}

(b) $X \in \mathcal{F}$ and

(c) given a set $T$ of the form $T = N \times Z$ with $Z$ compact and $\dim Z > 0$, there does not exist a homeomorphism of $T$ into $X$.

To show (a), note that the closure $\overline{X}$ of $X$ in the $(x, y)$-plane equals:

$$\overline{X} = \bigcup_{n=1}^{\infty} D_n \cup D_0,$$

where $D_0 = \{(x, y); x = 0, 0 \leq y \leq 1\}$ and $\overline{X}$ is a compact space. It differs from $X$ by the open interval $D_0 - \{(p) \cup (q)\}$, which is an $F_\sigma$ in $\overline{X}$. Therefore (a) holds.

To show (b), we shall prove that the assumption $X \in \mathcal{F}$ leads to a contradiction.

Suppose, that $X \in \mathcal{F}$. Then:

(1) There exists a homeomorphism $h: X \to Y$ of $X$ into a compact space $Y$, such that $\dim [Y - h(X)] < \dim X = 1$.

Since the intervals $D_n$ are disjoint, the sets $h(D_n); n = 1, 2, \ldots$ form a sequence of disjoint continua (even arcs) in the compact space $Y$. Thus, there exists a subsequence $\{k\}$ of natural numbers, such that the continua $h(D_k)$ converge to a continuum $E = \lim_{k \to \infty} h(D_k)$.

Now, it is easily seen that

$$\text{(b1)} \quad \text{the diameter } \delta(E) > 0.$$

Indeed, if $E$ were to reduce to a point $\bar{p}$, there would be, for the endpoints $p_k$ and $q_k$ of $D_k$: $p_k \to \bar{p}, h(p_k) \to \bar{p} = h(p)$ and $q_k \to \bar{p}, h(q_k) \to \bar{p} = h(q)$ which is impossible, since $h$ is a one-to-one mapping.

We also have

$$\text{(b2)} \quad E \cap h(D_n) = 0 \quad \text{for every } n = 1, 2, \ldots$$

since otherwise there would exist a number $n_0$, a point $r \in D_{n_0}$, a subsequence $\{j\}$ of $\{k\}$ and points $r_j \in D_j$ such that $\lim_{j \to \infty} r_j = r$ which is impossible by the definition of the intervals $D_n$. (No interval $D_n$ contains a limit point of a sequence of points belonging to intervals $D_n$ for $n \neq n_0$.)

By (b1), $E$ is a continuum containing more than one point and therefore $\dim E \geq 1$. But by (b2) we have $E \subset Y - h(\bigcup_{n=1}^{\infty} D_n)$. Hence by $h(X) = h(\bigcup_{n=1}^{\infty} D_n) \cup (h(p)) \cup (h(q))$ we have $\dim [Y - h(X)] \geq 1$ which contradicts (1).

Thus (b) holds. It remains to show (c). For this purpose suppose, to the contrary, that there would exist a compact set $Z$ with $\dim Z > 0$ \footnote{This follows from [2, p. 110, Theorem 4]. It can also be derived from [5, p. 11, (9, 11)].}
such that the set \( T = N \times Z \) has a topological image \( f(T) \) in \( X \). Since \( \dim Z > 0 \), the compact set \( Z \) contains a continuum \( C \) which does not reduce to one point.\(^{10}\) Therefore the set \( T = N \times Z \) would contain the set \( N \times C \) which consists of \( 2^N \) disjoint continua \( C_i \) and we could write \( N \times C = \bigcup_{i \in N} C_i \). The image \( f(C_i) \) of every \( C_i \) would be a continuum contained in \( X \).\(^{11}\) But \( X \) is a union of a denumerable sequence of closed sets. Hence, by a theorem of Sierpinski\(^{12}\) the set \( f(C_i) \) has to be contained in one and only one, interval \( D_n = D_n(t) \), \( n = 1, 2, \ldots \). Thus \( f(C_i) \) would be, for every \( \xi \), a closed interval contained in an interval \( D_n(t) \). Now for \( \xi' \neq \xi'' \) the intervals \( f(C_{\xi'}) \) and \( f(C_{\xi''}) \) would be disjoint and therefore, there would exist a family of power \( 2^N \) of disjoint intervals contained in the set \( X \), which is impossible (since \( X \) is a union of a countable family of intervals and two points). Therefore (c) also holds.

**Remark.** As noted in footnote 8, the proof of (1) is a consequence of Theorem 1, p. 31 of [3]. This theorem concerns finite-dimensional spaces. Now it is easily seen that (1) follows also from the fact that

(2) If \( X \in \mathfrak{A} \), then there exists a compact space \( Y \) and a homeomorphism \( h: X \to Y \) such that \( h(X) = \bigcap_{i=1}^n G_i \) where \( G_i \) are open in \( Y \) and \( Y - h(X) = \bigcup_{i=1}^n \text{Fr}(G_i) \) with \( \dim \text{Fr}(G_i) < \dim X \).\(^{13}\)

Indeed, if \( X \in \mathfrak{A} \) then \( X \) can be represented in the form (1\(^o\)). Taking the closure \( \text{cl}(h(X)) \) of \( h(X) \) in \( Y \) and denoting this "new" set \( \text{cl}(h(X)) \) by \( Y \) and the "new" sets \( G_i \cap \text{cl}(h(X)) \) by \( G_i \), it is easy to verify that (2) holds without any assumption of finite-dimensionality of \( X \).

**References**


*Technion, Israel Institute of Technology, Haifa*

\(^{10}\) See [2, p. 130]; also [4, p. 278].
\(^{11}\) Evidently \( f(C_i) \) contains more than one point.
\(^{13}\) See [2, p. 113].
\(^{12}\) The proof is analogous to that of the necessity of Theorem 1, p. 31 of [3].