CERTAIN PROBLEMS OF DIFFERENTIABLE IMBEDDING

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1. The differentiable imbedding of the complex projective spaces has been studied by many authors [1; 3; 4; 5]. In this note we shall deal with the nonimbeddability of the submanifolds of a complex projective space. It was studied in [6] in a particular case.

We denote by $P_n(c)$ the complex projective space of complex dimension $n$. Let $V_{2n-2}$ be a differentiable compact orientable submanifold of $P_n(c)$ corresponding to a cohomology class $v \in H^2(P_n(c), \mathbb{Z})$. Then the Pontrjagin class of $V_{2n-2}$ is determined as follows [2]:

\begin{align*}
(1.1) & \quad j : V_{2n-2} \to P_n(c), \\
(1.2) & \quad 1 + p_1(V_{2n-2}) + p_2(V_{2n-2}) + \cdots \nonumber \\
& \quad = j^*[(1 + p_1(P_n(c)) + p_2(P_n(c)) + \cdots)(1 + v^2)^{-1}] 
\end{align*}

where $p_i$ denotes the Pontrjagin class of the dimension $4i$. We put as follows:

\begin{align*}
(1.3) & \quad p = \sum_{k=0}(-1)^k \tilde{p}_k = \prod_{\alpha} (1 - \gamma_{\alpha}), \\
(1.4) & \quad \tilde{p} = \sum_{k=0} \tilde{p}_k = \prod_{\alpha} (1 - \gamma_{\alpha})^{-1}, \\
(1.5) & \quad p \cdot \tilde{p} = 1.
\end{align*}

In the case of $P_n(c)$ we have

\begin{align*}
(1.6) & \quad p = (1 - g_n)^{n+1}, \quad g_n \in H^2(P_n(c), \mathbb{Z}), \\
(1.7) & \quad \tilde{p} = (1 - g_n)^{-n-1}.
\end{align*}

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Hence we have from (1.2), (1.4) and (1.7)

\[ P(V_{2m-2}) = j^*[(1 - v^2)(1 - g_n)^{-n-1}] \]

When \( v = \lambda g \), where \( \lambda \) denotes some integer, we have from (1.8)

\[ P(V_{2m-2}) = j^*[(1 - \lambda^2 g_n^2)(1 - g_n)^{-n-1}] \]
\[ = j^* \left[ 1 + (n + 1 - \frac{2}{\lambda}) g_n + \cdots \right. \]
\[ + \left. \frac{(n+1) \cdots (n+r)}{r!} - \frac{(n+1) \cdots (n+r-1)}{(r-1)!} \frac{\lambda^2}{g_n} \right]^{2r} \]
\[ + \cdots \]

Meanwhile, if a compact orientable differentiable manifold \( V_m \) is differentiably imbedded in an \((m+q)\)-dimensional euclidean space \( E_{m+q} \), it must be that

\[ P_k = 0, \quad 2k \geq q + 1. \]

Moreover, \( P_k = 0 \) if \( 2k \geq q \), since if \( q = 2k \), \( P_k = E^2 \), where \( E \) is the Euler class of the normal bundle, and \( E = 0 \) in such a case.

Let us examine the term of the highest dimension in (1.10). When \( n = 2m+1 \), it is

\[ \left\{ \frac{(2m + 1) \cdots (3m + 1)}{m!} - \frac{(2m + 1) \cdots (3m - 2)}{(m - 1)!} \frac{\lambda^2}{g_n} \right\}^{2m} \]

When \( m \geq 1 \) the quantity (1.12) never vanishes. Hence we have from (1.11)

**Theorem 1.** Any compact orientable differentiable submanifold \( V_{4m} (m \geq 1) \) of \( P_{2m+1}(\mathbb{C}) \) cannot be differentiably imbedded in the \( E_{4m} \).

When \( n = 2m \), the term of the highest dimension in (1.10) is as follows:

\[ \left\{ \frac{(2m + 1) \cdots (3m - 1)}{(m - 1)!} - \frac{(2m + 1) \cdots (3m - 2)}{(m - 2)!} \frac{\lambda^2}{g_n^{2m-2}} \right\}^{2m} \]

If \( m \neq 3 \), the quantity (1.13) does not vanish. Hence we have

**Theorem 2.** Any compact orientable differentiable submanifold \( V_{4m-2} (m \neq 3) \) of \( P_{2m}(\mathbb{C}) \) cannot be differentiably imbedded in the \( E_{6m-4} \).

The exceptional cases for above theorem are those where \( m = 3 \) and
\( \lambda = \pm 2 \). Even those submanifolds cannot be imbedded in the \( E_{12} \), because the coefficient of \( g^2 \) in (1.10) does not vanish.

2. The following theorem is available for our purpose:

**Theorem (Atiyah-Hirzebruch [4]).** Let \( X_{2n} \) be a differentiable manifold and its integral Stiefel-Whitney class \( w_2 \) be zero. If there exists a Chern character \( z \in \text{Ch}(X_{2n}) \) whose \( s(z) \) is odd, then such an \( X_{2n} \) cannot be imbedded in the sphere whose dimension is \( 4n - 2\alpha(n) \), where \( \alpha(n) \) denotes the number of 1 in the diadic expansion of \( n \). In particular, if there exists a cohomology class \( d \in H^2(X_{2n}, \mathbb{Z}) \) whose \( d^n(x_{2n}) \) is odd, then \( X_{2n} \) cannot be imbedded in the sphere whose dimension is \( 4n - 2\alpha(n) \).

Let \( V_{2n-2} \) be a submanifold of \( P_n(c) \), i.e.,

\[
(2.1) \quad j: V_{2n-2} \to P_n(c)
\]

and

\[
(2.2) \quad v = \lambda g_n,
\]

where \( v \) denotes the cohomology class corresponding to \( V_{2n-2} \). Then we have

\[
(2.3) \quad d = j^*(g_n),
\]

\[
(2.4) \quad d^{n-1}[V_{2n-2}] = j^*[g_n^{n-1}][V_{2n-2}] = (v g_n^{n-1})[P_n(c)] = \lambda(g_n)[P_n(c)] = \lambda.
\]

Such a \( V_{2n-2} \) satisfies the condition \( w_3 = 0 \), because \( w(P_n(c)) \) lacks the terms of odd dimension and the normal bundle of \( V_{2n-2} \) has the same property. Hence we have from (2.4)

**Theorem 3.** If \( \lambda \) is odd, the submanifold of \( P_n(c) \), corresponding to \( \lambda g_n \), cannot be differentiably imbedded in the \( E_{4n-4-2\alpha(n-1)} \).

**References**


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