CERTAIN PROBLEMS OF DIFFERENTIABLE IMBEDDING

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1. The differentiable imbedding of the complex projective spaces has been studied by many authors [1; 3; 4; 5]. In this note we shall deal with the nonimbeddability of the submanifolds of a complex projective space. It was studied in [6] in a particular case.

We denote by \( P_n(c) \) the complex projective space of complex dimension \( n \). Let \( V_{2n-2} \) be a differentiable compact orientable submanifold of \( P_n(c) \) corresponding to a cohomology class \( v \in H^2(P_n(c), \mathbb{Z}) \). Then the Pontrjagin class of \( V_{2n-2} \) is determined as follows [2]:

\[
\begin{align*}
(1.1) & \quad j : V_{2n-2} \to P_n(c), \\
(1.2) & \quad 1 + p_1(V_{2n-2}) + p_2(V_{2n-2}) + \cdots = j^*[1 + p_1(P_n(c)) + p_2(P_n(c)) + \cdots ](1 + v^2)^{-1}j
\end{align*}
\]

where \( p_i \) denotes the Pontrjagin class of the dimension \( 4i \). We put as follows:

\[
\begin{align*}
(1.3) & \quad p = \sum_{k=0}^{\infty} (-1)^k \bar{p}_k = \prod_{a} (1 - \gamma_a), \\
(1.4) & \quad \bar{p} = \sum_{k=0}^{\infty} \bar{p}_k = \prod_{a} (1 - \gamma_a)^{-1}, \\
(1.5) & \quad p \cdot \bar{p} = 1.
\end{align*}
\]

In the case of \( P_n(c) \) we have

\[
\begin{align*}
(1.6) & \quad p = (1 - g_n)^{n+1}, \quad g_n \in H^2(P_n(c), \mathbb{Z}), \\
(1.7) & \quad \bar{p} = (1 - g_n)^{-n-1}.
\end{align*}
\]

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Hence we have from (1.2), (1.4) and (1.7)

\[ \tilde{f}(V_{2n-2}) = j^*[(1 - v^2)(1 - g_\ast)^{-n}] \]

When \( v = \lambda g \), where \( \lambda \) denotes some integer, we have from (1.8)

\[ \tilde{f}(V_{2n-2}) = j^*[(1 - \lambda g_\ast)(1 - g_\ast)^{-n}] \]
\[ = j^* \left[ 1 + (n + 1 - \lambda^2)g_\ast + \cdots \right] \]
\[ + \left\{ \frac{(n+1) \cdots (n+r)}{r!} - \frac{(n+1) \cdots (n+r-1)}{(r-1)!} \lambda^2 \right\} g_\ast^{2r} \]
\[ + \cdots \].

Meanwhile, if a compact orientable differentiable manifold \( V_m \) is differentiably imbedded in an \((m+q)\)-dimensional euclidean space \( E_{m+q} \), it must be that

\[ \tilde{\psi}_k = 0, \quad 2k \geq q + 1. \]

Moreover, \( \tilde{\psi}_k = 0 \) if \( 2k \geq q \), since if \( q = 2k, \tilde{\psi}_k = E^2 \), where \( E \) is the Euler class of the normal bundle, and \( E = 0 \) in such a case.

Let us examine the term of the highest dimension in (1.10). When \( n = 2m+1 \), it is

\[ \left\{ \frac{(2m+2) \cdots (3m+1)}{m!} - \frac{(2m+2) \cdots (3m)}{(m-1)!} \lambda^2 \right\} g_{2m+1}. \]

When \( m > 1 \) the quantity (1.12) never vanishes. Hence we have from (1.11)

**Theorem 1.** Any compact orientable differentiable submanifold \( V_{4m} (m > 1) \) of \( P_{2m+1}(c) \) cannot be differentiably imbedded in the \( E_{4m} \).

When \( n = 2m \), the term of the highest dimension in (1.10) is as follows:

\[ \left\{ \frac{(2m+1) \cdots (3m-1)}{m-1!} - \frac{(2m+1) \cdots (3m-2)}{(m-2)!} \lambda^2 \right\} g_{2m-2}. \]

If \( m \neq 3 \), the quantity (1.13) does not vanish. Hence we have

**Theorem 2.** Any compact orientable differentiable submanifold \( V_{4m-2} (m \neq 3) \) of \( P_{2m}(c) \) cannot be differentiably imbedded in the \( E_{4m-4} \).

The exceptional cases for above theorem are those where \( m = 3 \) and
λ = ± 2. Even those submanifolds cannot be imbedded in the $E_{12}$, because the coefficient of $g^2$ in (1.10) does not vanish.

2. The following theorem is available for our purpose:

**Theorem (Atiyah-Hirzebruch [4]).** Let $X_{2n}$ be a differentiable manifold and its integral Stiefel-Whitney class $w_3$ be zero. If there exists a Chern character $s \in Ch(X_{2n})$ whose $s(z)$ is odd, then such an $X_{2n}$ cannot be imbedded in the sphere whose dimension is $4n - 2\alpha(n)$, where $\alpha(n)$ denotes the number of 1 in the diadic expansion of $n$. In particular, if there exists a cohomology class $d \in H^2(X_{2n},\mathbb{Z})$ whose $d^n(x_{2n})$ is odd, then $X_{2n}$ cannot be imbedded in the sphere whose dimension is $4n - 2\alpha(n)$.

Let $V_{2n-2}$ be a submanifold of $P_n(c)$, i.e.,

$$j: V_{2n-2} \to P_n(c)$$

and

$$\nu = \lambda g_n,$$

where $\nu$ denotes the cohomology class corresponding to $V_{2n-2}$. Then we have

$$d = j^*(g_n),$$

$$d^{n-1}[V_{2n-2}] = j^*[g_n^{n-1}][V_{2n-2}] = (vg_n^{n-1})[P_n(c)] = \lambda(g_n)[P_n(c)] = \lambda.$$

Such a $V_{2n-2}$ satisfies the condition $w_3 = 0$, because $w(P_n(c))$ lacks the terms of odd dimension and the normal bundle of $V_{2n-2}$ has the same property. Hence we have from (2.4)

**Theorem 3.** If $\lambda$ is odd, the submanifold of $P_n(c)$, corresponding to $\lambda g_n$, cannot be differentiably imbedded in the $E_{4n-4-2\alpha(n-1)}$.

**References**


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