A SUPPLEMENT TO PARKER’S “REMARKS ON BALANCED INCOMPLETE BLOCK DESIGNS”

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1. E. T. Parker proved [2] the following theorem.

Let $D$ be a balanced incomplete block design (BIBD) with parameters $(v, b, k, r, \lambda) = (2x + 2, 4x + 2, x + 1, 2x + 1, x)$, where $x$ is a positive even integer. Then (i) any two blocks of $D$ have at least one common element, and (ii) no two blocks of $D$ are the same subset.

Parker showed further that conclusion (i) need not hold when $x$ is odd. He concludes his remarks saying, “Whether conclusion (ii) holds for odd $x$ appears less easy to decide.” It is the purpose of this note to show that conclusion (ii) holds also for $x$ odd.

The result of this note is established using formulae of Bose and Bush [1] connecting parameters of orthogonal arrays $(\lambda s^2, k, s, 2)$:

(a) $\sum_{i=1}^{s} n_i = \lambda s^2 - 1$,
(b) $\sum_{i=1}^{s} i n_i = k(\lambda s - 1)$,
(c) $\sum_{i=1}^{s} i(i-1)n_i = k(k-1)(\lambda - 1)$,

where $n_i$ denotes the number of columns which have $i$ coincidences with any chosen fixed column of the array, $\lambda$ is the frequency of each ordered pair in like columns of each pair of distinct rows of the array, $k$ is the number of rows, and $s$ is the number of values the elements of the array can take on.

2. On conclusion (ii) of Parker’s Theorem. Conclusion (ii) holds for any positive integer $x$.

Proof. Parker shows that the existence of BIBD with parameters $(2x + 2, 4x + 2, x + 1, 2x + 1, x)$, $x$ a positive integer, implies the existence of an orthogonal array with $s = 2, \lambda = x + 1, k = 2x + 2$. If conclusion (ii) does not hold then this array would have to include at least two identical columns. This means that equations (a), (b), and (c) would have to hold with $n_{2\lambda} \geq 1$. For $k = 2\lambda, s = 2$ equations (a), (b) and (c) reduce to

$$\sum_{\tilde{s} \leq 2\lambda} n_{\tilde{s}} = 4\lambda - 1$$
$$\sum_{\tilde{s} \leq 2\lambda} i n_{\tilde{s}} = 2\lambda(2\lambda - 1)$$

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\[ \sum_{0 \leq i \leq 2\lambda} i(i - 1) = 2\lambda(2\lambda - 1)(\lambda - 1). \]

Hence, the average number of coincidences, say, \( \bar{n} = 2\lambda(2\lambda - 1)/(4\lambda - 1). \)
Furthermore \[ \sum_{0 \leq i \leq 2\lambda} (i - \bar{n})^2 = 2\lambda^2(2\lambda - 1)/(4\lambda - 1). \]
If \( n_{2\lambda} = 1 \) then the contribution of this term alone to the total sum of squares equals \( 16\lambda^4/(4\lambda - 1)^2 \) which is clearly impossible.

REFERENCES


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