ON FINITELY ADDITIVE SET FUNCTIONS

S. P. LLOYD

1. Introduction. Let $\mathcal{F}$ be a $\sigma$-field of subsets of a set $X$, and let $\text{ba}(X, \mathcal{F})$ be the Banach space of all bounded real valued finitely additive set functions on $\mathcal{F}$, with total variation norm. Let $\text{ca}(X, \mathcal{F})$ be the members of $\text{ba}(X, \mathcal{F})$ which are countably additive, and let $\text{pf}(X, \mathcal{F})$ be the members of $\text{ba}(X, \mathcal{F})$ which are purely finitely additive. Then $\text{ba}(X, \mathcal{F}) = \text{ca}(X, \mathcal{F}) \oplus \text{pf}(X, \mathcal{F})$ [1, p. 163; 2].

Let $X^*$ be the Boolean space of the Boolean algebra $\mathcal{B}$. We assume without essential loss of generality that $\mathcal{F}$ separates the points of $X$, so that $X$ may be regarded as a dense subset of $X^*$. Space $\text{ba}(X, \mathcal{F})$ is isometrically isomorphic to the space $C^*(X^*)$ of bounded real valued signed Baire measures on $X^*$. Solving a problem raised in [2], we characterize $\text{pf}(X, \mathcal{F})$ as the set functions whose correspondents in $C^*(X^*)$ live on a Baire $F_\sigma$ set of first category in $X^*$; such a set is necessarily contained in $X^* - X$, as it turns out. Further results are given for the case where each point of $X$ is a measurable set.

2. Preliminaries. Let $B = B(X, \mathcal{F})$ be the Banach algebra of bounded real valued $\mathcal{F}$ measurable functions on $X$, with point-wise operations and supremum norm. We represent the conjugate space as $B^* = \text{ba}(X, \mathcal{F})$.

Let $X^* \subset B^*$ be the set of homomorphisms of $B$ onto the reals, with the relative $B^*/B$ (weak*) topology. Then $B$ is isometrically algebraically isomorphic to the algebra $C(X^*)$ of all continuous real valued functions on compact Hausdorff $X^*$. If $f \in C(X^*)$ corresponds to $f \in B$ then $\hat{f}$ is simply the unique continuous extension of $f$ to all of $X^*$, regarding $X$ as a (dense) subset of $X^*$. In particular, to each $E \in \mathcal{F}$ there corresponds the open closure $\overline{E}$ of $E$ in $X^*$, and each open and closed set in $X^*$ is of the form $\overline{E}$ for some $E \in \mathcal{F}$. The sets $\overline{E}$, $E \in \mathcal{F}$, are a base for the topology of $X^*$. The correspondence $E \leftrightarrow \overline{E}$ preserves finite set operations, and $\overline{E} = E \cap X$, $E \in \mathcal{F}$.

The smallest $\sigma$-field containing all $\overline{E}$, $E \in \mathcal{F}$, is the $\sigma$-field $\mathcal{B}$ of Baire subsets of $X^*$. We represent $C^*(X^*)$ as the space of bounded real valued signed Baire measures on $X^*$. If $\lambda \in C^*(X^*)$ corresponds to $\lambda \in B^*$ then $\lambda$ is determined by the values $\lambda(\overline{E}) = \lambda(E)$, $E \in \mathcal{F}$.

3. Representation.

Theorem 1. Let $F \subset X^*$ be a closed Baire set. Then $F$ is nowhere dense if and only if $F \subset X^* - X$. 

Received by the editors July 18, 1962.

701
Proof. Suppose $F$ is a closed subset of $X^* - X$. If $F$ contained a neighborhood then $F$ would contain points of $X$, since $X$ is dense. Thus $F$ is nowhere dense. Conversely, let $F$ be a closed nowhere dense Baire set. Then $F$ is a $G_δ$: $F = \bigcap_i G_n$, $G_n$ open [4, p. 221]. For each $n$ there is a cover of the compact set $G'_n$ by open and closed sets disjoint from $F$. Hence there is an open and closed set $E_n$, $E_n \subseteq \mathcal{F}$, such that $F \subseteq E_n \subseteq G_n$. Thus $F = \bigcap_i E_n$, $F \cap X = \bigcap_i E_n$. If $E = \bigcap_i E_n \subseteq \mathcal{F}$ were nonempty then $F$ would contain the neighborhood $E$, and it follows that $F \subseteq X^* - X$. (In general, $E$ is the open and closed interior of the closed Baire set $F$; cf. Theorem 4.11 of [2].)

The following result is Theorem 4.14 of [2].

Theorem 2. Suppose $\lambda \in \mathcal{B}^*$. Then for $\lambda \in \mathcal{C}(X, \mathfrak{F})$ to hold it is necessary and sufficient that $\lambda(F) = 0$ for each closed nowhere dense Baire set $F$.

Theorems 1 and 2 show that in the sense of outer measure, $X$ is thick in $X^*$ relative to any $\lambda$ such that $\lambda \in \mathcal{C}(X, \mathfrak{F})$ [4, p. 74]. As we shall see, however, this does not imply that the members of $\mathcal{C}(X, \mathfrak{F})$ have support $X$ in $X^*$.

We denote by $\mathfrak{C}$ the class of first category Baire $F_σ$ sets, i.e., countable unions of closed nowhere dense Baire sets. For $\lambda \in \mathcal{C}(X^*)$ and $A \in \mathfrak{A}$ we denote by $\check{\lambda}_A$ the measure $\check{\lambda}_A(E) = \check{\lambda}(E \cap A)$, $E \in \mathfrak{A}$.

Theorem 3. Suppose $\lambda \in \mathcal{B}^*$. Then for $\lambda \in \mathcal{P}(X, \mathfrak{F})$ to hold it is necessary and sufficient that $\lambda = \check{\lambda}_c$ for some $C_0 \subseteq \mathfrak{C}$.

Proof. Sufficiency is a consequence of Theorem 4.16 of [2]. Conversely, suppose $\lambda \geq 0$, $\lambda \in \mathcal{P}(X, \mathfrak{F})$. Let number $m$ be defined by

$$m = \sup_{C \in \mathfrak{C}} \check{\lambda}(C).$$

Let $\{C_n\}$ be a sequence in $\mathfrak{C}$ such that $\lim \check{\lambda}(C_n) = m$. Then $C_0 = \bigcup_i C_n$ is a member of $\mathfrak{C}$ and $\check{\lambda}(C_0) = m$. Consider the measure $\mu = \check{\lambda} - \check{\lambda}_{C_0}$. We have $\check{\mu}(C) = 0$, $C \in \mathfrak{C}$, clearly, so that $\mu \in \mathcal{C}(X, \mathfrak{F})$, from Theorem 2. Since $0 \leq \mu \leq \lambda$ and $\lambda \in \mathcal{P}(X, \mathfrak{F})$, we must have $\mu = 0$, and the theorem follows.

We assume from now on that each point of $X$ is measurable; $\{x\} \in \mathfrak{F}$ for each $x \in X$. Then each point $x \in X$ is an open and closed set in $X^*$, $X$ is open and $Y = X^* - X$ is closed in $X^*$. Suppose $\lambda = \lambda_1 + \lambda_2$, $\lambda_1 \in \mathcal{C}(X, \mathfrak{F})$, $\lambda_2 \in \mathcal{P}(X, \mathfrak{F})$, $\lambda_1$, $\lambda_2 \geq 0$. Denote by $\check{\lambda}$ the regular Borel extension of $\check{\lambda}$. Then with $\check{\lambda}_3(E) = \check{\lambda}(E \cap X)$, $\check{\lambda}_4(E) = \check{\lambda}(E \cap Y)$, Borel $E \subseteq X^*$, we have $\check{\lambda} = \check{\lambda}_3 + \check{\lambda}_4$. It is clear from Theo-
rems 1 and 2 that \( \lambda_3 \in \mathcal{C}(X, \mathcal{F}) \); let us see what this measure is. Since \( \lambda_3 \) is regular, we have

\[
\|\lambda_3\| = \lambda_3(X) = \sup_{C \in X} \lambda_3(C),
\]

where the supremum is over all compact subsets of \( X \) in \( X^* \). But the only compact subsets of \( X \) in \( X^* \) are the finite subsets of \( X \). As in the proof of Theorem 3, we find a countable subset \( A_1 \) of \( X \) such that \( \lambda_3(E) = \lambda_3(E \cap A_1) \), Borel \( E \subset X^* \). Thus \( \lambda_1 \) is the purely atomic part of \( \lambda_1 \). The atomless part of \( \lambda_1 \) is contained in \( \lambda_4 \), living on \( Y \); this is \( \lambda_5 = \lambda_4 - \lambda_2 \). Let \( \lambda_*, \lambda^* \) denote the inner and outer measures induced by \( \lambda \). Then \( \lambda_*(Y) = \|\lambda_3\|, \lambda^*(Y) = \|\lambda_3\| + \|\lambda_5\| \), whence the Borel set \( Y \) is not \( \lambda^* \) measurable when \( \lambda_5 \neq 0 \). A fortiori, \( \lambda \) is not completion regular when \( \lambda_5 \neq 0 [4, p. 230] \).

To summarize, \( \lambda \in B^* \) has the decomposition \( \lambda = \lambda_* + \lambda_2 + \lambda_5 \), where \( \lambda \in \mathcal{P}(F \setminus \mathcal{F}) \) corresponds to \( \lambda_2 \) living on a Baire \( F \), subset of \( X^* - X \), where the purely atomic part \( \lambda_3 \) corresponds to \( \lambda_3 \) living on a countable subset of \( X \) in \( X^* \), and where the atomless countably additive part \( \lambda_5 \) corresponds to \( \lambda_5 \) which lives on \( X^* - X \) and assigns measure 0 to every Baire \( F \) subset of \( X^* - X \). (It is easily seen that \( X^* - X \) is a Baire set if and only if \( X \) is countable. There are no atomless countably additive measures in this case, of course.)

4. Measurable cardinals. The above results throw some light on, but do not solve, the problem of the existence of measurable cardinals [3, Chapter 12]. Let \( X \) be a discrete space of cardinal \( |X| \), and let \( \mathcal{F} \) be the field of all subsets of \( X \). Then \( X^* = \beta X \) is the Stone-Čech compactification of \( X \). For each \( x \in X^* \) denote by \( \lambda_x \in \mathcal{C}(X, \mathcal{F}) \) the set function corresponding to the unit point measure at \( x: \lambda_x(E) = \chi_x(E), E \in \mathcal{F} \), where \( \chi_x \) is the characteristic function of \( E \). The cardinal \( |X| \) is said to be measurable if \( \lambda_x \in \mathcal{C}(X, \mathcal{F}) \) for some \( x \in X^* - X \).

Denote by \( \Gamma = \bigcup_{C \in C} C \) the union of all closed Baire subsets of \( X^* - X \), and define \( \Delta = (X^* - X) - \Gamma \).

**Theorem 4.** The cardinal \( |X| \) is measurable if and only if \( \Delta \) is nonempty.

**Proof.** By Theorems 1 and 2, \( \lambda_x \in \mathcal{C}(X, \mathcal{F}) \) if and only if \( x \) is contained in no closed Baire subset of \( X^* - X \).

A point \( x \in \Delta \) has the property that \( x \) is an interior point of every closed Baire set containing it. Dually, if \( x \in \Delta \) is disjoint from an open Baire set \( U \) then \( x \) is disjoint from the open closure \( \overline{U} \) of \( U \). It is easy
to show that every neighborhood of \( x \in \Delta \) meets \( X \) in a set of measurable cardinal.

*Added in proof.* Theorem 4 is not new. If \( X \) is any completely regular space, then \( \beta X - \nu X \) is the union of all Baire subsets of \( \beta X \) disjoint from \( X \); cf. [3, 119].

**References**


Bell Telephone Laboratories