WEAKLY COMPACT $B^\ell$-ALGEBRAS

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1. A complex Banach algebra $A$ is a compact (weakly compact) algebra if its left and right regular representations consist of compact (weakly compact) operators. Let $E$ be any subset of $A$ and denote by $E_i$ and $E_r$ the left and right annihilators of $E$. $A$ is an annihilator algebra if $A_i = (0) = A_r$, $I, I_r \neq (0)$ for each proper closed left ideal $I$ and $J, J_r \neq (0)$ for each proper closed right ideal $J$.

In [6, Theorem 1], it was shown that a semi-simple compact algebra is an annihilator algebra. The first main result of the present paper (Theorem 2.1) is that a semi-simple annihilator algebra is a weakly compact algebra. Thus if $C, A, W$ denote respectively the class of all semi-simple compact algebras, all semi-simple annihilator algebras and all weakly compact algebras, we have $C \subset A \subset W$.

§3 is devoted to the structure theory of weakly compact $B^\ell$-algebras begun in [7]. A Banach algebra $A$ is a $B^\ell$-algebra if, given $a \in A$, there exists $a^\ell \neq 0$ in $A$ such that

$$\|a^\ell\| \|a\| = \|(a^\ell a)^n\|^{1/n}, \quad n = 1, 2, 3, \ldots$$

In their study of weakly compact $B^\ast$-algebras Ogasawara and Yoshi-naga [4] obtained the following structure theorem:

**Theorem.** The following statements are equivalent for a $B^\ast$-algebra $A$:

1. $A$ is a weakly compact algebra;
2. $A$ is the $B^\ast(\infty)$-sum of $C^\ast$-algebras, each of which consists of the set of all compact operators on a Hilbert space.

The following result was obtained in [7, Theorem 3.1]:

**Theorem.** A Banach algebra $A$ is the algebra $F(X)$ of all uniform limits of operators of finite rank on a reflexive Banach space $X$ if and only if $A$ is a simple, weakly compact $B^\ell$-algebra with minimal ideals.

Making use of this result and our present Theorem 2.1, we now obtain the following more general result:

**Theorem 3.4.** The following statements are equivalent:

1. $A$ is a weakly compact $B^\ell$-algebra with a dense socle;
2. $A$ is the $B(\infty)$-sum of $B^\ell$-algebras, each of which is the algebra of all uniform limits of operators of finite rank on a reflexive Banach space.

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We note that every $B^*$-algebra is a $B^\sharp$-algebra and that a weakly compact $B^*$-algebra automatically has a dense socle [4, p. 15], so that Theorem 3.4 includes the result of Ogasawara and Yoshinaga.

2. Theorem 2.1. A semi-simple annihilator algebra $A$ is a weakly compact algebra.

Proof. Let $Ae$, $e^2 = e$, be a minimal left ideal of $A$. Then $Ae$ is a reflexive Banach space since it is also a minimal left ideal of the simple annihilator algebra $(AeA)^\perp$ [1, Theorem 13]. Let $a \in A$; then [3, p. 483, Corollary 3] right multiplication by $ae$ is a weakly compact mapping of $A$ into $Ae$, and a fortiori, of $A$ into $A$. Then by [3, p. 484, Theorem 5], right multiplication by a socle element is weakly compact. Since the socle is dense [1, Theorem 4], it follows [3, p. 483, Corollary 4] that any $x \in A$ is a right (and similarly left) weakly compact element.

3. In this section we prove a structure theorem for weakly compact $B^\sharp$-algebras.

Lemma 3.1. Let $A$ be a semi-simple Banach algebra with a dense socle. Then every maximal regular left ideal $M$ of $A$ has a nonzero right annihilator.

Proof. If $\{Ae_a\}_{a \in A}$ denotes the set of all the minimal left ideals of $A$, then there exists $\alpha_0 \in \Omega$ such that $Ae_{\alpha_0} \subseteq M$. Further, $M \cap Ae_{\alpha_0} = (0)$ and $M \oplus Ae_{\alpha_0} = A$. Since $M$ is a regular left ideal of $A$, there exists $j \in A$ such that $xj - x \in M$ for every $x \in A$. For some $\alpha_0 \in A$ and $m_0 \in M$, we have $jm = m_0 + ae_{\alpha_0} = 0 \in 0$. Let $m$ be an arbitrary element of $M$; then $mj - m \in M$ and $mj - ma_{\alpha_0} = mm_0 \in M$, from which it follows that $m - ma_{\alpha_0} \in M$, and therefore $ma_{\alpha_0} \in M$. However, $ma_{\alpha_0} \in A$, since $Ae_{\alpha_0} = A$, and since $m$ is arbitrary in $M$, the lemma is proved.

Lemma 3.2. Let $A$ be a $B^\sharp$-algebra with a dense socle. If $| \cdot |$ is any norm in $A$ with $|a| \leq ||a||$ for each $a \in A$, then $| \cdot | = || \cdot ||$.

Proof. Suppose that $j \in A$ and $j$ has no left reverse. We show that there exists $a \neq 0$ such that $ja = a$. In fact, let $J = [yj - y : y \in A]$; then $J$ is a regular left ideal of $A$ which is proper since $j \notin J$. Now $J$ is contained in a maximal regular left ideal $M$ and by Lemma 3.1, there exists $a \in A$, $a \neq 0$ such that $Ja = (0)$, i.e. such that $yj a - y a = 0$ for all $y \in A$; i.e., $A(ja - a) = (0)$. Since $A$, being a $B^\sharp$-algebra is semi-simple, $A_a = (0)$ from which it follows that $ja = a$. The conclusion now follows exactly as in [2, Theorems 3 and 4].
Lemma 3.3. A semi-simple Banach algebra $A$ with a dense socle (or with the annihilator property) is the completion of the direct join of all its minimal closed two-sided ideals.

This is essentially Theorem 6 of Bonsall and Goldie [1], under the hypothesis that $A$ be an annihilator algebra. The annihilator property implies that $A$ has a dense socle and this, together with the semi-simplicity of $A$, is all that is required to prove the theorem.

Definition. Let $\{A_\alpha\}_{\alpha \in \Omega}$ denote a set of Banach algebras. The $B(\infty)$-sum of the $A_\alpha$ is the Banach algebra $A$ consisting of all the functions $f(\cdot)$ defined on $\Omega$ with $f(\alpha) \in A_\alpha$ for each $\alpha \in \Omega$ and such that, given $\epsilon > 0$, there is a finite subset $\alpha_1, \alpha_2, \ldots, \alpha_n$ of $\Omega$ such that $\|f(\alpha)\| < \epsilon$ for $\alpha \neq \alpha_1, \alpha_2, \ldots, \alpha_n$. We define the algebraic operations in $A$ in the obvious way, e.g. $(f+g)(\alpha) = f(\alpha) + g(\alpha)$, etc. and define the norm by $\|f(\cdot)\| = \sup_{\alpha \in \Omega} \|f(\alpha)\|$.

We now state our second main result:

Theorem 3.4. The following statements are equivalent:

1. $A$ is a weakly compact $B^\ast$-algebra with a dense socle.
2. $A$ is the $B(\infty)$-sum of $B^\ast$-algebras $A_\alpha$, $\alpha \in \Omega$, each of which is the algebra of all uniform limits of operators of finite rank on a reflexive Banach space.

Proof. (1) $\Rightarrow$ (2). Bonsall [2, Theorem 6] has shown that if $A$ is a $B^\ast$-annihilator algebra, then $A$ is isomorphic and isometric to the $B(\infty)$-sum of its minimal closed two-sided ideals $A_\alpha$. For the case of a $B^\ast$-algebra with a dense socle, Bonsall's proof applies almost word for word if Lemmas 3.2 and 3.3 are borne in mind.

That each $A_\alpha$ is a weakly compact algebra in its own right is clear and that $A_\alpha$ is simple follows readily from a routine argument which depends essentially on the fact that $A$ is semi-simple. Thus each $A_\alpha$ is a simple, weakly compact, $B^\ast$-algebra with minimal ideals. (That $A_\alpha$ contains a minimal left ideal of its own follows from the fact that $A_\alpha$ contains a minimal left ideal of $A$ which is also a minimal left ideal of $A_\alpha$.) Hence by [7, Theorem 3.1], each $A_\alpha$ is the algebra of all uniform limits of operators of finite rank on a reflexive Banach space.

(2) $\Rightarrow$ (1). Each $A_\alpha$, being the algebra of all uniform limits of operators of finite rank on a reflexive Banach space, is a $B^\ast$ annihilator algebra [2, Theorem 2]. Since a $B^\ast$-algebra is semi-simple, the $B(\infty)$-sum of the $A_\alpha$ is, by a result of Rickart's [8, p. 107], an annihilator algebra. That the $B(\infty)$-sum of an arbitrary class of $B^\ast$-algebras is a $B^\ast$-algebra is proved in [5, Lemma 4.7]. Thus the $B(\infty)$-sum $A$ of the $A_\alpha$ is a semi-simple annihilator algebra. From this it follows that...
A has a dense socle and by Theorem 2.1, \( A \) is weakly compact. This concludes the proof.

**Corollary.** A B*-algebra is an annihilator algebra if and only if it has a dense socle and is a weakly compact algebra.

It is to be noted that one-handed weak complete continuity is enough to prove \((1) \Rightarrow (2)\). In fact, a very slight modification of the proofs of [7, Lemma 3.1 and Theorem 3.1] shows that a simple, left weakly compact B*-algebra with minimal ideals is isomorphic and isometric to the algebra \( F(X) \) of all uniform limits of operators of finite rank on a reflexive Banach space \( X \). Since the \( B(\infty) \)-sum of the \( A_\sigma \) in Theorem 3.4 is an annihilator algebra and weakly compact, we obtain the following:

**Theorem 3.5.** A right weakly compact B*-algebra with a dense socle is a weakly compact algebra.

**References**


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