ON ODD PERFECT NUMBERS. II

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One of the oldest unsolved mathematical problems is the following one: Are there odd perfect numbers? So many interesting necessary conditions for an odd integer to be perfect have been found out. A bibliography of previous work is given by McCarthy [5]. Throughout this paper \( n \) denotes an odd perfect number.

The following results have been proved in [1] and [2] respectively:

(i) \( 1/2 < \sum_{p|n} (1/p) < 2 \log (\pi/2) (\sim 0.903) \),

(ii) \( n \) must be of the form \( 12t+1 \) or \( 36t+9 \).

The bounds for \( \sum_{p|n} (1/p) \) given in [1] have been improved in [3] as

(a) \[
\frac{\log 2}{5 \log \left(\frac{5}{4}\right)} < \sum_{p|n} \frac{1}{p} < \log 2 + \frac{1}{338},
\]

if \( n \) is of the form \( 12t + 1 \),

(b) \[
\frac{1}{3} + \frac{\log 4}{5 \log \frac{5}{4}} < \sum_{p|n} \frac{1}{p} < \log \frac{18}{13} + \frac{53}{150},
\]

if \( n \) is of the form \( 36t + 9 \).

The object of this paper is to further improve the bounds for \( \sum_{p|n} (1/p) \).

The following Tables I and II give numerical values for the bounds obtained in [3] and the bounds obtained in this paper respectively.

| Table I |
|-----------------|-----------------|--------------------|
|                | Lower bound     | Upper bound        | Difference        |
| (a)            | .621            | .696               | .075              |
| (b)            | .591            | .679               | .088              |

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It can be easily seen from Table II that (a) if \( n \) is of the form \( 12t+1 \), .644 < \( \sum_{p|n} (1/p) \) < .693, which is of range .049, a one-third cut in the length of the interval of \([3]\) and (b) if \( n \) is of the form \( 36t+9 \), .596 < \( \sum_{p|n} (1/p) \) < .674, which is of range .078, an improvement over \([3]\) of about 12 per cent.

**Table II**

<table>
<thead>
<tr>
<th>(α)</th>
<th>.644</th>
<th>.679</th>
<th>.035</th>
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<tbody>
<tr>
<td>(β)</td>
<td>.657</td>
<td>.693</td>
<td>.036</td>
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<tr>
<td>(γ)</td>
<td>.596</td>
<td>.674</td>
<td>.078</td>
</tr>
<tr>
<td>(δ)</td>
<td>.600</td>
<td>.662</td>
<td>.062</td>
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The bounds obtained are given by the following:

**Theorem.** (α) If \( n \) is of the form \( 12t+1 \) and \( 5|n \),

\[
\frac{1}{5} + \frac{1}{7} + \frac{48}{35} \frac{\log 11}{11 \log 10} < \sum_{p|n} \frac{1}{p} < \frac{1}{5} + \frac{1}{2738} + \log \frac{50}{31} .
\]

(β) If \( n \) is of the form \( 12t+1 \) and \( 5\n \),

\[
\frac{1}{7} + \frac{12}{7} \frac{\log 11}{11 \log 10} < \sum_{p|n} \frac{1}{p} < \log 2 .
\]

(γ) If \( n \) is of the form \( 36t+9 \) and \( 5|n \),

\[
\frac{1}{3} + \frac{1}{5} + \frac{16}{15} \frac{\log 17}{17 \log 16} < \sum_{p|n} \frac{1}{p} < \frac{1}{3} + \frac{1}{5} + \frac{1}{13} + \log \frac{65}{61} .
\]
(δ) If \( n \) is of the form \( 36\ell + 9 \) and \( 5 \mid n \),

\[
\frac{1}{3} + \frac{\log 3}{3} < \sum_{\ell | n} \frac{1}{\ell} \leq \frac{1}{3} + \frac{1}{338} + \frac{\log 18}{13}.
\]

**Proof.** We prove this theorem in various cases and in each case one can see that either the lower bound or the upper bound for \( \sum_{\ell | n} (1/\ell) \) as stated in this theorem is further improved.

Euler proved that \( n \) must be of the form \( p_0^{\alpha_0} \cdot x^2 \), where \( p_0 \) is a prime of the form \( 4\lambda + 1 \), \( \alpha_0 \) is of the form \( 4\mu + 1 \), \( \lambda > 1 \) and \( (p_0, x) = 1 \). Hence we can write \( n = p_0^{\alpha_0} \cdot p_1^{\alpha_1} \cdot p_2^{\alpha_2} \cdots p_k^{\alpha_k} \), where \( \alpha_r \) is even for \( 1 \leq r \leq k \). We shall suppose as we may do without loss of generality that \( p_1 < p_2 < \cdots < p_k \). Let \( \sigma(n) \) denote the sum of the positive divisors of \( n \). Since \( n \) is a perfect number, we have \( \sigma(n) = 2n \), from which it can easily be seen that

\[
\frac{1}{3} + \frac{1}{338} + \frac{\log 18}{13}.
\]

Therefore

\[
\log 2 < - \sum_{\ell = 0}^{k} \log \left( 1 - \frac{1}{\ell} \right)
\]

(B)

\[
= \sum_{\ell = 0}^{k} \frac{1}{\ell} + \frac{1}{2} \sum_{\ell = 0}^{k} \frac{1}{\ell^2} + \frac{1}{3} \sum_{\ell = 0}^{k} \frac{1}{\ell^3} + \cdots.
\]

Taking logarithms of both sides of (A) and expressing them in series, we have

\[
\log 2 = \sum_{\ell = 0}^{k} \sum_{i=1}^{\infty} \left[ \frac{1}{i p^i_r} - \frac{1}{i p^i_r (\alpha+1)} \right].
\]

(C)

\[
= \sum_{\ell = 0}^{k} \frac{1}{\ell} + \sum_{\ell = 0}^{k} \sum_{i=1}^{\infty} \left[ \frac{1}{(i + 1) p^{i+1}_r} - \frac{1}{i p^i_r (\alpha+1)} \right].
\]

(a) Suppose \( n \) is of the form \( 12\ell + 1 \). In this case it has been proved in [3, p. 134] that \( p_0 \) is of the form \( 12N + 1 \) and hence \( p_0 \geq 13 \).
(a) If \( 5 \mid n \) and \( 7 \mid n \), then \( p_1 = 5 \), \( p_2 = 7 \) and \( p_r \geq 11 \) for \( 3 \leq r \leq k \). Now \( \alpha_3 \geq 4 \) for, if \( \alpha_3 = 2 \), then \( \sigma(p_2^{\alpha_3}) = 3.19 \) and since \( \sigma(n) = 2n \) it would follow that \( 3 \mid n \), which cannot hold.

From (B), we get that

\[
\log 2 < - \log \left(1 - \frac{1}{3}\right) - \log \left(1 - \frac{1}{7}\right) + \frac{1}{p_0} + \frac{1}{2} \frac{1}{11} \frac{1}{p_0} + \ldots \\
+ \sum_{r=3}^{k} \frac{1}{p_r} + \frac{1}{11} \sum_{r=3}^{k} \frac{1}{p_r} + \frac{1}{3} \frac{1}{11^2} \sum_{r=3}^{k} \frac{1}{p_r} + \ldots \\
= \log \frac{5}{4} + \log \frac{7}{6} + \left(\frac{1}{p_0} + \sum_{r=3}^{k} \frac{1}{p_r}\right) \\
\cdot \left(1 + \frac{1}{2} \frac{1}{11} + \frac{1}{3} \frac{1}{11^2} + \ldots \right) \\
= \log \frac{5}{4} + \log \frac{7}{6} \\
+ 11 \log \frac{11}{10} \left[\sum_{r=3}^{k} \frac{1}{p_r} - \frac{1}{5} - \frac{1}{7}\right]
\]

therefore

\[
\sum_{r=0}^{k} \frac{1}{p_r} > \frac{1}{5} + \frac{1}{7} + \frac{\log 48}{35} \\
\cdot \frac{1}{11 \log \frac{11}{10}}
\]

Also from (C), we get that

\[
\log 2 = \sum_{r=0}^{k} \frac{1}{p_r} + \sum_{r=1}^{\infty} \sum_{i=1}^{\infty} \left[\frac{1}{(i + 1) p_r^{i+1}} - \frac{1}{i p_r^{(\alpha_r+1)i}}\right] \\
+ \left(\frac{1}{2} \frac{1}{p_0^{\alpha_0+1}} - \frac{1}{p_0}\right) + \sum_{i=2}^{\infty} \left[\frac{1}{(i + 1) p_0^{i+1}} - \frac{1}{i p_0^{(\alpha_0+1)i}}\right].
\]

Now each term in the brackets of the second summation is positive, since \( \alpha_r \geq 2 \) for \( r > 0 \), and hence the second sum is positive. Similarly
the fourth sum is also positive, and \(1/2\rho_0^2-1/\rho_0^{\alpha+1} \geq -1/2\rho_0^2 \geq -1/338\), since \(\alpha_0 \geq 1\) and \(\rho_0 \geq 13\). Therefore

\[
\log 2 > \sum_{r=0}^{k} \frac{1}{p_r} + \sum_{i=1}^{\infty} \left[ \frac{1}{(i+1)5^{i+1}} - \frac{1}{i(5^i)} \right] + \sum_{i=1}^{\infty} \left[ \frac{1}{(i+1)7^{i+1}} - \frac{1}{i(7^i)} \right] - \frac{1}{338}, \text{ since } \alpha_1 \geq 2 \text{ and } \alpha_3 \geq 4
\]

\[
= \sum_{r=0}^{k} \frac{1}{p_r} - \log \left(1 - \frac{1}{5}\right) - \frac{1}{5} + \log \left(1 - \frac{1}{5^3}\right) - \log \left(1 - \frac{1}{7}\right) - \frac{1}{7} + \log \left(1 - \frac{1}{7^3}\right) - \frac{1}{338}.
\]

Therefore

\[
\sum_{r=0}^{k} \frac{1}{p_r} < \frac{1}{5} + \frac{1}{7} + \frac{1}{338} + \log \frac{50}{31} \cdot \frac{2401}{31}.
\]

\(\text{(a}_{18}\text{)}\)

\[
< \frac{1}{5} + \frac{1}{7} + \frac{1}{338} + \log \frac{50}{31}.
\]

Hence by \((\text{a}_{11})\) and \((\text{a}_{18})\), \((\alpha)\) follows in this case.

\((\alpha_2)\) If \(5 \mid n\) and \(7 \nmid n\), then \(p_1 = 5\) and \(p_r \geq 11\) for \(2 \leq r \leq k\). Since \(\alpha_0\) is odd \((1 + p_0)\mid \sigma(\rho_0^n)\) and hence \((1 + p_0)/2 \mid n\) since \(\sigma(n) = 2n\). Now \(p_0\) is not 13, since otherwise it would follow that \(7 \mid n\), which is not the case. Since \(p_0\) is of the form \(12N + 1\), \(p_0 \geq 37\).

From (B) as in \((\text{a}_1)\) we can get that

\[
\log 2 < \log \frac{5}{4} + 11 \log \frac{11}{10} \left[ \frac{1}{\rho_0} + \sum_{r=2}^{k} \frac{1}{p_r} \right]
\]

\[
= \log \frac{5}{4} + 11 \log \frac{11}{10} \left[ \frac{1}{\rho_0} - \frac{1}{5} \right];
\]

therefore

\[
\sum_{r=0}^{k} \frac{1}{p_r} > \frac{1}{5} + \frac{11 \log \frac{8}{5}}{10 \log \frac{11}{10}}.
\]

\(\text{(a}_{21}\text{)}\)

\[
> \frac{1}{5} + \frac{1}{7} + \frac{11 \log \frac{48}{35}}{10 \log \frac{11}{10}}.
\]
From (D), arguing in a similar way as in (a₁), we can get that
\[
\log 2 > \sum_{r=0}^{k} \frac{1}{p_r} + \sum_{i=1}^{\infty} \left[ \frac{1}{(i + 1)5^{i+1}} - \frac{1}{i(5^i)i} \right] - \frac{1}{2(37)^2}
\]
\[
= \sum_{r=0}^{k} \frac{1}{p_r} - \log \left( 1 - \frac{1}{5} \right) - \frac{1}{5} + \log \left( 1 - \frac{1}{5^5} \right) - \frac{1}{2738}.
\]
Therefore
\[
(a_{3R}) \quad \sum_{r=0}^{k} \frac{1}{p_r} < \frac{1}{5} + \frac{1}{2738} + \log \frac{50}{31}.
\]
Hence by (a₃L) and (a₃R), (α) follows in this case.

Thus (α) is proved.

(a₃) If 5 \mid n and 7 \mid n, then \(p_1 = 7\) and \(p_r \geq 11\) for \(2 \leq r \leq k\). Now \(\alpha_1 \geq 4\) as we have seen in (a₁) that \(\alpha_1 \neq 2\). From (B) as in the case (a₁), we get that
\[
\log 2 < \log \frac{7}{6} + 11 \log \frac{11}{10} \sum_{r=0}^{k} \frac{1}{p_r} - \frac{1}{7}.
\]
Therefore
\[
(a_{3L}) \quad \sum_{r=0}^{k} \frac{1}{p_r} > \frac{1}{7} + \frac{\log \frac{12}{7}}{11 \log \frac{11}{10}}.
\]
From (D), arguing in a similar way as in (a₁), we get that
\[
\log 2 > \sum_{r=0}^{k} \frac{1}{p_r} - \log \left( 1 - \frac{1}{7} \right) - \frac{1}{7} + \log \left( 1 - \frac{1}{7^5} \right) - \frac{1}{338}.
\]
Therefore
\[
(a_{3R}) \quad \sum_{r=0}^{k} \frac{1}{p_r} < \frac{1}{7} + \frac{1}{338} + \log \frac{4802}{2801} < \log 2.
\]
Hence by (a₃L) and (a₃R), (β) follows in this case.

(a₄) If 5 \mid n and 7 \mid n, then \(p_r \geq 11\) for \(1 \leq r \leq k\).

From (B) as in (a₁), we get that \(\log 2 < 11 \log 11/10 \cdot \sum_{r=0}^{k} 1/p_r\).
Therefore

\[(a_{44}) \sum_{r=0}^{k} \frac{1}{p_r} > \frac{\log 2}{11 \log \frac{11}{10}} > \frac{1}{7} + \frac{\log 12}{11 \log \frac{11}{10}}.\]

Now as in \((a_2)\), we see that \((1+p_0)/2|n\). Let \(p\) be any prime dividing \((1+p_0)/2\), then \(\pi | n\) and hence \(\pi = p_j\) for some \(j\) satisfying \(1 \leq j \leq k\).

From \((D)\), arguing in a similar way as in \((a_1)\), we see that

\[
\log 2 > \sum_{r=0}^{k} \frac{1}{p_r} + \sum_{i=1}^{\infty} \left[ \frac{1}{(i+1)p_{i+1}^j} - \frac{1}{i(p_0)^i} \right] + \left( \frac{1}{2p_0^2} - \frac{1}{p_0^4} \right), \quad \text{since } \alpha_0 \geq 1 \text{ and } \alpha_j \geq 2
\]

\[
> \sum_{r=0}^{k} \frac{1}{p_r} + \left( \frac{1}{2p_0^2} - \frac{1}{p_0^4} \right) - \frac{1}{2p_0^2}
\]

\[
> \sum_{r=0}^{k} \frac{1}{p_r}, \quad \text{since } 11 \leq p_j \leq \frac{1+p_0}{2}.
\]

Therefore

\[(a_{44b}) \sum_{r=0}^{k} \frac{1}{p_r} < \log 2.\]

Hence by \((a_{44})\) and \((a_{44b})\), \((\beta)\) follows in this case.

Thus \((\beta)\) is proved.

(b) Suppose \(n\) is of the form \(36t+9\). Since \(3|n\), \(p_1 = 3\).

\((b_1)\) If \(5|n\), then \(7|n\) in virtue of the result that \(3.5.7\) does not divide \(n\) (proved in Kühnel \([4]\)).

\((b_1.a)\) If at least one of \(11\) and \(13\) divides \(n\), then obviously

\[
\sum_{r=0}^{k} \frac{1}{p_r} > \frac{1}{3} + \frac{1}{5} + \frac{1}{13} > \frac{1}{3} + \frac{1}{5} + \frac{16}{17 \log 16}.
\]

Otherwise,

\((b_1.a)\) \(p_r \geq 17\) for \(2 \leq r \leq k\), if \(p_0 = 5\); or

\((b_1.a)\) \(p_r \geq 17\) for \(3 \leq r \leq k\), if \(p_2 = 5\). In this particular case \(p_0\) is also \(\geq 17\), since \(p_0 \neq 5\) and \(p_0\) is not \(13\), since we are in the case where neither \(11\) nor \(13\) divides \(n\).
In both the cases (b1.2) and (b1.3), from (B) as in (a1) we can get that

\[
\log 2 < \log \frac{3}{2} + \log \frac{5}{4} + 17 \log \left[ \sum_{r=0}^{k} \frac{1}{p_r} - \frac{1}{3} - \frac{1}{5} \right].
\]

Hence in any case under (b1), we have that

\[
(b_{1L}) \quad \sum_{r=0}^{k} \frac{1}{p_r} > \frac{1}{3} + \frac{1}{5} + \frac{\log 16}{17 \log 16}.
\]

For the upper bound, the proof for the cases (1) \( p_0 \neq 5 \) and (2) \( p_0 = 5 \)
and \( \alpha_1 \geq 4 \) are omitted as they are similar to the previous proofs. In both these cases we easily verify that the bound obtained is less than the bound obtained for the case \( p_0 = 5 \) and \( \alpha_1 = 2 \).

For this case, since \( \alpha_1 = 2 \), \( \sigma(p_0^{\alpha_1}) = 13 \), so \( 13 \mid n \).

We then obtain from (D), arguing in a similar way as in (a1),

\[
\log 2 > \sum_{r=0}^{k} \frac{1}{p_r} + \sum_{i=1}^{\infty} \left[ \frac{1}{(i+1)3^{i+1}} - \frac{1}{i(3^3)^i} \right]
\]

\[
+ \sum_{i=1}^{\infty} \left[ \frac{1}{(i+1)5^{i+1}} - \frac{1}{i(5^5)^i} \right] + \sum_{i=1}^{\infty} \left[ \frac{1}{(i+1)13^{i+1}} - \frac{1}{i(13^3)^i} \right]
\]

\[
= \sum_{r=0}^{k} \frac{1}{p_r} - \log \left( 1 - \frac{1}{3} \right) - \frac{1}{3} + \log \left( 1 - \frac{1}{3^3} \right) - \log \left( 1 - \frac{1}{5} \right) - \frac{1}{5}
\]

\[
+ \log \left( 1 - \frac{1}{5^3} \right) - \log \left( 1 - \frac{1}{13} \right) - \frac{1}{13} + \log \left( 1 - \frac{1}{13^3} \right).
\]

Therefore

\[
(b_{1R}) \quad \sum_{r=0}^{k} \frac{1}{p_r} < \frac{1}{3} + \frac{1}{5} + \frac{1}{13} + \log \frac{65}{61}.
\]

Hence by (b1L) and (b1R), (\( \gamma \)) follows.

(b2) If \( 5 \mid n \), then \( p_r \geq 7 \) for \( 2 \leq r \leq k \) and \( p_0 \geq 13 \).

From (B) as in (a1) we get that

\[
\log 2 < \log \frac{3}{2} + 7 \log \frac{7}{6} \left[ \sum_{r=0}^{k} \frac{1}{p_r} - \frac{1}{3} \right];
\]

therefore

\[
(b_{2L}) \quad \sum_{r=0}^{k} \frac{1}{p_r} > \frac{1}{3} + \log \frac{4}{3} \sqrt[7]{\log \frac{7}{6}}.
\]
From (D), arguing in a similar way as in (a1), we get that

$$\log 2 > \sum_{r=0}^{k} \frac{1}{p_r} - \log \left(1 - \frac{1}{3}\right) - \frac{1}{3} + \log \left(1 - \frac{1}{3^2}\right) - \frac{1}{338}.$$

Therefore

$$(b_{2n}) \quad \sum_{r=0}^{k} \frac{1}{p_r} < \frac{1}{3} + \frac{1}{338} + \log \frac{18}{13}.$$

Hence by (b_{2L}) and (b_{2R}), (δ) follows.
Thus the proof of the theorem is complete.

REFERENCES


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