RATIONAL NORMAL MATRICES SATISFYING
THE INCIDENCE EQUATION

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1. Introduction. The construction of a finite projective plane with
n points on a line is equivalent to determining an integral \(v \times v\)
matrix \(A\) \((v = n^2 + n + 1)\) satisfying

\[ A' A = A A = n I + S \]

where \(S\) is the matrix all of whose coordinates are 1. The Bruck-
Ryser Theorem [2] asserts that when \(n \equiv 1\) or 2 (mod 4), a necessary
and sufficient condition for the existence of a rational \(A\) satisfying
(1) is that \(n\) be a sum of two squares. In [1] Albert gives a construc-
tion for such a rational \(A\). The purpose of this note is to give a simpler
construction.

2. Notation. We denote \(I_r\) the \(r \times r\) identity matrix, \(S_r\) the \(r \times r\)
matrix all of whose entries are 1, \(e_r\) the \(1 \times r\) matrix (row vector) all
of whose entries are 1. If \(m = \frac{1}{2} (v-1) = \frac{1}{2} n (n+1)\), we denote

\[ E = \begin{pmatrix} 0 & I_m \\ -I_m & 0 \end{pmatrix} \]

so that \(E'E = I_{2m}\) and \(E + E' = 0\). We suppose \(n = a^2 + b^2\) and let
\(H = a I_{2m} + b E\) so that \(H'H = n I_{2m}\). Let

\[ P = \begin{pmatrix} 0 & \frac{1}{n} e_{2m} H \\ \frac{1}{n} e_{2m} H & H \end{pmatrix} \]

then clearly \(P\) is a rational \(v \times v\) matrix satisfying \(P'P = nI_v + S_v\).
Moreover

\[ P'P = nI_v + xx \quad \text{where} \quad x = \left( \frac{1}{n} e_{2m} H \right) \]

is a \(1 \times v\) row vector.

3. A rational solution. To obtain a rational solution of (1) it suffices
to find a rational orthogonal \(v \times v\) matrix \(T\) such that if \(A = PT\) then
\(T' A A = n I_v + S_v\). But \(T' A A = T' P P T = T(n I_v + xx) T = n I_v + (x T)(x T)\).

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hence it suffices to find a rational orthogonal $T$ satisfying $(xT)(xT)^T = S_e$. Since $S_e = e_v e_v^T$, we must find a rational $T$ satisfying $xT = e_v$.

Noting that $x^T x = e_v^T e_v = v$, i.e., the vectors $x$ and $e_v$ have the same length, it is apparent that the symmetry with respect to the plane perpendicular to the vector $x - e_v$ is the required transformation:

$$T = I_v - \frac{2t(x - e)(x - e)}{(x - e)^T(x - e)}.$$

REFERENCES


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