RATIONAL NORMAL MATRICES SATISFYING THE INCIDENCE EQUATION

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1. Introduction. The construction of a finite projective plane with \( n \) points on a line is equivalent to determining an integral \( v \times v \) matrix \( A \) \((v=n^2+n+1)\) satisfying

\[
A' A = A A = nI + S
\]

where \( S \) is the matrix all of whose coordinates are 1. The Bruck-Ryser Theorem [2] asserts that when \( n \equiv 1 \) or 2 \((\text{mod} \ 4)\), a necessary and sufficient condition for the existence of a rational \( A \) satisfying (1) is that \( n \) be a sum of two squares. In [1] Albert gives a construction for such a rational \( A \). The purpose of this note is to give a simpler construction.

2. Notation. We denote \( I_r \) the \( r \times r \) identity matrix, \( S_r \) the \( r \times r \) matrix all of whose entries are 1, \( e_r \) the \( 1 \times r \) matrix (row vector) all of whose entries are 1. If \( m = \frac{1}{2}(v-1) = \frac{1}{2}n(n+1) \), we denote

\[
E = \begin{pmatrix}
0 & I_m \\
-I_m & 0
\end{pmatrix}
\]

so that \( E'E = I_{2m} \) and \( E + I' = 0 \). We suppose \( n = a^2 + b^2 \) and let \( H = aI_{2m} + bE \) so that \( H'H = nI_{2m} \). Let

\[
P = \begin{pmatrix}
0 & \frac{1}{n} e_{2m} H \\
\frac{1}{n} e_{2m} H & H
\end{pmatrix}
\]

then clearly \( P \) is a rational \( v \times v \) matrix satisfying \( P'P = nI_v + S_v \). Moreover

\[
'PP = nI_v + 'xx \quad \text{where} \quad x = \left( n \frac{1}{n} e_{2m} H \right)
\]

is a \( 1 \times v \) row vector.

3. A rational solution. To obtain a rational solution of (1) it suffices to find a rational orthogonal \( v \times v \) matrix \( T \) such that if \( A = PT \) then \( 'AA = nI_v + S_v \). But \( 'AA = 'TPPT = 'T(nI_v + 'xx)T = nI_v + ('xT)(xT) \);

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hence it suffices to find a rational orthogonal $T$ satisfying $^t(xT)(xT) = S_e$. Since $S_e = e^t e$, we must find a rational $T$ satisfying $xT = e$. Noting that $x^t x = e^t e = v$, i.e., the vectors $x$ and $e$ have the same length, it is apparent that the symmetry with respect to the plane perpendicular to the vector $x - e$, is the required transformation:

$$T = I_v - \frac{2t(x - e)(x - e)}{(x - e)^t(x - e)}.$$

**References**


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