ON THE HYPERBOLIC CAPACITY AND CONFORMAL MAPPING\(^1\)

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1. Let \( E \) be a compact set in \( D = \{ |z| < 1 \} \). Tsuji \([6]\) has introduced the hyperbolic capacity of \( E \) which can be defined by

\[
\text{caph } E = \lim_{{n \to \infty}} \max_{{E}} \prod_{{\mu = 0}}^{{n}} \prod_{{\nu \mu}} \left\lvert \frac{z_\mu - z_\nu}{1 - \bar{z}_\mu z_\nu} \right\rvert^{1/n(n+1)}.
\]

Also,

\[
\min_{{f}} \max_{{z \in E}} \left\lvert f(z) \right\rvert^{1/n} \to \text{caph } E
\]
as \( n \to \infty \) where the minimum is taken over all functions

\[
f(z) = \prod_{{r=1}}^{{n}} e^{i\alpha_r(z - z_r)/(1 - \bar{z}_r z)} \quad (\alpha_r \text{ real, } |z_r| < 1).
\]

We shall first obtain another formula for \( \text{caph } E \). Leja \([1]\) has proved an analogous formula for the capacity of a plane set.

**Lemma.** Let \( E \) be a compact set in \( D \). For each \( n = 1, 2, \ldots \) choose \( n+1 \) points \( z_0, \ldots, z_n \) in \( E \) such that

\[
\prod_{{\mu = 0}}^{{n}} \prod_{{\nu \mu}} \left\lvert z_\mu - z_\nu \right\lvert / \left\lvert 1 - \bar{z}_\mu z_\mu \right\rvert
\]
becomes maximal. Numerate these points so that

\[
A_n = \prod_{{r=1}}^{{n}} \left\lvert z_0 - z_r \right\lvert / \left\lvert 1 - \bar{z}_r z_0 \right\rvert
\]

\[
= \min_{{\mu}} \prod_{{\nu \mu}} \left\lvert z_\mu - z_\nu \right\lvert / \left\lvert 1 - \bar{z}_\mu z_\mu \right\rvert.
\]

If

\[
f_n(z) = \prod_{{r=1}}^{{n}} \frac{1 - \bar{z}_r}{1 - z_r} \frac{z - z_r}{1 - \bar{z}_r z}
\]

then

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\[ \max_{z \in E} |f_n(z)| = A_n, \]

and, as \( n \to \infty \)

\[ A_n^{1/n} \to \text{caph } E. \]

**Proof.** For \( |z_1| < 1, |z_2| < 1 \) we write \([z_1, z_2] = |z_1 - z_2| / |1 - z_1 z_2|\).

Let \( z \in E \). Comparing the system \( z, z_1, \ldots, z_n \) of points in \( E \) with the maximal system \( z_0, z_1, \ldots, z_n \) we see that

\[
\begin{align*}
1 \cdot [z, z_1] & \cdots [z, z_n] & 1 \cdot [z_0, z_1] & \cdots [z_0, z_n] \\
[z_1, z] \cdot 1 & \cdots [z_1, z_n] & [z_1, z_0] \cdot 1 & \cdots [z_1, z_n] \\
& \vdots & & \vdots \\
[z_n, z] & [z_n, z_1] & \cdots 1 & [z_n, z_0] [z_n, z_1] & \cdots 1.
\end{align*}
\]

Hence \( |f_n(z)| \leq A_n \), with equality for \( z = z_0 \), which proves (6). Since \( f_n \) has the form (3) it follows that \( \min f \max_{z \in E} |f(z)| \leq A_n \). Therefore by (2)

\[
\liminf_{n \to \infty} A_n^{1/n} \geq \text{caph } E.
\]

On the other hand, (4) implies

\[
A_n^{n+1} \leq \prod_{\mu=0}^{n} \prod_{\nu=0}^{n} [z_{\mu}, z_{\nu}].
\]

Hence (1) shows that \( \limsup_{n \to \infty} A_n^{1/n} \leq \text{caph } E \), and the Lemma follows.

2. Let \( E \) be a compact set in \( D = \{ |z| < 1 \} \). Then \( D \setminus E \) is an open set of which exactly one component region \( G \) has \( \{ |z| = 1 \} \) as part of the boundary. I shall give an elementary proof of the following theorem.

**Theorem 1.** Let \( \rho = \text{caph } E > 0 \). If \( f_n(z) \) is defined by (5) then

\[ g(z) = \lim_{n \to \infty} f_n(z)^{1/n} \]

exists locally uniformly in \( H = G \cup \{ 1 \leq |z| < r \} \) for some \( r > 1 \), and \( g(z) \) is the smallest function satisfying

1. \( g(z) \) is locally analytic\(^2\) and of single-valued modulus in \( H \),
2. \( |g(z)| = 1 \) for \( |z| = 1 \),
3. \( 1 \geq |g(z)| \geq \rho \) for \( z \in G \),

that is, if \( h(z) \) also satisfies these three conditions then \( |g(z)| \leq |h(z)| \) for \( z \in G \).

\(^2\) This means that \( g(z) \) is analytic on the universal covering surface of \( H \).
Furthermore, \( g(1) = 1 \) and

\[
\int_0^{2\pi} \frac{d\arg g(e^{i\theta})}{\theta} = 2\pi.
\]

If \( \xi \) is a boundary point of \( G \) that lies on a continuum contained in \( E \) then \( |g(z)| \to \rho \) for \( z \to \xi \), \( z \in G \).

Finally, if \( E \) is a continuum then \( \rho > 0 \), and \( w = g(z) \) maps \( G \) conformally and one-to-one onto \( \{ \rho < |w| < 1 \} \).

**Remarks.** Let

\[
\omega(z) = \log(\rho^{-1}|g(z)|) / \log \rho^{-1}.
\]

Then Theorem 1 shows that \( \omega(z) \) is the smallest function satisfying

(a') \( \omega(z) \) is single-valued and harmonic in \( H \),

(b') \( \omega(z) = 1 \) for \( |z| = 1 \),

(c') \( 1 \leq \omega(z) \leq 0 \) for \( z \in G \),

that is, if \( \nu(z) \) also satisfies these three conditions then \( \omega(z) \leq \nu(z) \).

If the boundary of \( G \) consists of a finite number of nondegenerate continua then \( \omega(z) = 0 \) on the boundary points of \( G \) that lie in \( D \). Hence \( \omega(z) \) is the harmonic measure of \( \{|z| = 1\} \) with respect to \( G \). By (8),

\[
1/\log \rho^{-1} = \frac{1}{2\pi} \int_0^{2\pi} \frac{\partial}{\partial r} \omega(r e^{i\theta}) \left| \frac{dr}{d\theta} \right| d\theta.
\]

Of course, we could have started with the harmonic measure and then proved (9). But the method applied here is simpler and more constructive. It does not use set-functions, the solvability of the Dirichlet problem or the Riemann mapping theorem. The existence of a function that maps a doubly-connected region onto an annulus is established (see also [4]).

The following proof uses (with some simplifications) the method of extremal points developed by Leja [1; 2; 3].

**Proof.** a. The Lagrange interpolation formula shows that

\[
\sum_{\mu=0}^{n} \left( \prod_{\nu=0}^{n} \frac{z - z_{\mu}}{z_{\nu} - z_{\mu}} \cdot \prod_{r=1}^{n} (1 - \bar{z}_{\mu} z_r) \right) = \prod_{r=1}^{n} (1 - \bar{z}_r z).
\]

Hence

\[
\max_{\mu} \left( \prod_{\nu=0}^{n} \left| \frac{z - z_{\mu}}{z_{\nu} - z_{\mu}} \right| \cdot \prod_{r=1}^{n} \left| \frac{1 - \bar{z}_{\mu} z_r}{1 - \bar{z}_r z} \right| \right) \geq \frac{1}{n + 1}.
\]

Let \( g(z) = \min_{z_1, z_2 \in E} \left| \frac{z - z_1}{z - z_2} \right| \) (for \( z \in G \)). Since \( E \subset \{|z| \leq a\} \) for some \( a < 1 \) it follows that
\[
\max_{\mu} \left( \prod_{r=1}^{n} \left| \frac{z - z_r}{1 - \bar{z}_r z} \right| \right) \prod_{r \neq \mu} \left| \frac{1 - \bar{z}_\mu}{z_\mu - z_r} \right| \geq \frac{(1 - a^2)q(z)}{2(n + 1)},
\]

and because of (4)

\[
|f_n(z)| \geq \frac{(1 - a^2)q(z)}{2(n + 1)}.
\]

We put \( r = 2/(1 + a) > 1 \). Since \( |z - z_r|/|1 - z_r z| \leq (r + a)/(1 - ar) < 4/(1 - a) \) for \( |z| \leq r \), (5) shows that

\[
|f_n(z)|^{1/n} < 4/(1 - a) \quad (|z| \leq r).
\]

b. Let \( H = G \cup \{ 1 \leq |z| \leq r \} \) and \( g_n(z) = f_n(z)^{1/n} \). The functions \( g_n(z) \) are locally analytic in \( H \), and \( |g_n(z)| \) is single-valued. By (11) and Montel's theorem we can find a sequence \( n_k \) such that \( g_{n_k}(z) \) converges locally uniformly in \( H \). Let \( g(z) \) be the limit function. Since by the Lemma \( A_{n}^{1/n} \rightarrow \rho \), inequality (10) implies \( |g(z)| \geq \rho \). Also \( |g(z)| = 1 \) for \( |z| = 1 \) so that \( g(z) \) satisfies (a), (b) and (c).

Let \( h(z) \) be any function satisfying these three conditions, and let \( z^* \) be a point in \( G \). Given \( \epsilon > 0 \) we choose a fixed \( k \) so large that \( |g_{n_k}(z^*)| > e^{-\epsilon} |g(z^*)| \). Since \( \rho > 0 \) we can take \( k \) so that also \( A_{n_k}^{1/n_k} < \rho e^{\epsilon} \). Then it follows from (6) that \( |g_{n_k}(z)| \leq \rho e^{\epsilon} \) for \( z \in E \). We choose analytic curves in \( G \) so near to \( E \) that their union \( C \) separates \( E \) from \( z^* \) and from \( \{ |z| = 1 \} \), and that \( |g_{n_k}(z)| \leq \rho e^{\epsilon^*} \) for \( z \in C \). Because \( |h(z)| \geq \rho \) for \( z \in G \),

\[
|g_{n_k}(z)|/|h(z)| \leq \rho e^{\epsilon^*}/\rho = e^{\epsilon^*}
\]

for \( z \in C \). Since the left side is \( = 1 \) for \( |z| = 1 \) it follows from the maximum principle that the inequality holds also for \( z = z^* \). Hence

\[
|g(z^*)| < e^\epsilon |g_{n_k}(z^*)| \leq e^{\epsilon^*} |h(z^*)|
\]

for every \( \epsilon > 0 \) and therefore \( |g(z^*)| \leq |h(z^*)| \).

Since \( f_n(1) = 1 \) we obtain \( g(1) = 1 \), and by the argument principle

\[
\int_0^{2\pi} d \arg g_n(e^{i\theta}) = \frac{1}{n} \int_0^{2\pi} d \arg f_n(e^{i\theta}) = 2\pi
\]

from which (8) follows.

If \( g_n(z) \) did not converge there would be a limit function \( h(z) \neq g(z) \) for some other subsequence of \( g_n \) as Montel's theorem shows. From what we have proved it follows that \( |h(z)| \geq |g(z)| \). Reversing the roles of \( g \) and \( h \) we also get \( |g(z)| \geq |h(z)| \). Hence \( |h(z)| = |g(z)| \), and \( g(1) = h(1) = 1 \) implies \( g(z) = h(z) \). Therefore \( g_n(z) \rightarrow g(z) \) as \( n \rightarrow \infty \).

c. We assume now that \( E \) is a continuum. We do not know yet that
\( \rho > 0 \). If \( \rho = 0 \) then \( \lim \) \( g_n(z) \) might not exist. In this case let \( g(z) \) be the limit function in \( H \) for some convergent subsequence of \( g_n \) which exists by (11). The region \( H \) is doubly connected, and every simply closed curve in \( H \) is homotopic either to a point or the unit circle. Therefore the functions \( g(z) \) and \( g_n(z) \) are single-valued because of (8) and (12). Let \( c \) be any point with \( p < |c| < 1 \). It follows from (12) that \( w = g_n(z) \) maps \( \{|z| = 1\} \) one-to-one onto \( \{|w| = 1\} \). Hence

\[
\frac{1}{2\pi i} \int_{|z|=1} \frac{g'_n(z)}{g_n(z) - c} \, dz = \frac{1}{2\pi i} \int_{|w|=1} \frac{1}{w - c} \, dw = 1.
\]

We choose analytic curves \( C_m \) \((m = 1, 2, \ldots)\) enclosing \( E \) so that the regions between \( C_m \) and \( \{|z| = 1\} \) approach \( G \). By the Lemma we can choose them so that \( |g_n(z)| < |c| \) on each \( C_m \) for sufficiently large \( n \). Then

\[
\frac{1}{2\pi i} \int_{C_m} \frac{g'_n(z)}{g_n(z) - c} \, dz = 0.
\]

Making \( n \to \infty \) we obtain from (13) and (14) that

\[
\frac{1}{2\pi i} \int_{|z|=1} \frac{g'(z)}{g(z) - c} \, dz - \frac{1}{2\pi i} \int_{C_m} \frac{g'(z)}{g(z) - c} \, dz = 1
\]

for all \( m \). Hence \( g(z) \) assumes the value \( c \) exactly once in \( G \). Therefore \( w = g(z) \) maps \( G \) one-to-one onto \( \{\rho < |w| < 1\} \).

Suppose that \( \rho = 0 \) were true. Then the inverse function \( \psi(w) \) of \( w = g(z) \) would be analytic and univalent in \( \{0 < |w| \leq 1\} \). Since \( |\psi(w)| < 1 \) it would follow that \( \psi(w) \) is bounded and univalent in \( \{|w| \leq 1\} \). Since \( |\psi(w)| = 1 \) for \( |w| = 1 \) this would imply that \( \psi(w) \) is a linear function and therefore \( E \) a point.

d. Let again \( E \) be arbitrary and let \( \xi \) be a boundary point lying on a continuum \( B \) that is contained in \( E \). Let \( g_0(z) \) be the function that maps the doubly-connected region between \( B \) and \( \{|z| = 1\} \) onto \( \{\rho_0 < |w| < 1\} \). Let \( \lambda \) be such that \( \rho_0^\lambda = \rho \). Then \( h(z) = g_0(z)^\lambda \) satisfies the conditions (a), (b) and (c) of Theorem 1. Hence, as we have already proved, \( \rho \leq |g(z)| \leq |h(z)| = |g_0(z)|^\lambda \). Since for simple topological reasons \( |g_0(z)| \to \rho_0 \) as \( z \to \xi \), \( z \in G \) it follows that \( |g(z)| \to \rho \).

3. We shall now prove an analogue to a result by Walsh [7] about the ordinary Green's function. We introduce the hyperbolic metric in the unit disk \( D \). A circle perpendicular to \( \{|z| = 1\} \) will be called a geodesic.

**Theorem 2.** Let \( E \) be a compact set in \( D \) with \( \operatorname{caph} E > 0 \), and let
g(z) be the function defined in Theorem 1. Let L(r) = \{z: |g(z)| = r\} (caph \ E < r < 1). Then at every point of L(r) the inner geodesic normal to L(r) intersects the hyperbolically convex hull K of E.

REMARKS. The hyperbolically convex hull of E is defined as the smallest closed set K ⊆ E that is convex in the hyperbolic metric in the sense that together with any two points also the geodesic segment between these two points belongs to K. The set L(r) is the union of a finite number of closed analytic curves which may have multiple points though. At the multiple points g'(z) vanishes. It is easy to see that all multiple points lie in K (see [8, p. 157]).

PROOF. With the notations of Theorem 1 let L_\infty(r) = \{z: |f_\infty(z)| = r^n\}. We first prove that the inner geodesic normal to L_\infty(r) at any z ∈ L_\infty(r) intersects K. Suppose this were false. Then \zeta \in \mathcal{K}. By a conformal mapping of the unit disk onto itself we can make \zeta = 0. Then the inner geodesic normal becomes a straight ray and is separated from K by a line. We may thus assume that K ⊆ \{Re z < 0\} and that the inner normal lies in \{Re z ≥ 0\}. Writing z_\infty = x_\infty + iy, we have x_\infty < 0. Hence

\[ \frac{d}{dz} \log f_\infty(z) \bigg|_{z=0} = \sum_{n=1}^{\infty} \frac{1 - |z_\infty|^2}{(z - z_\infty)(1 - \bar{z_\infty}z)} \bigg|_{z=0} = \sum_{n=1}^{\infty} \frac{1 - |z_\infty|^2}{|z_\infty|^2} (x_\infty + iy_\infty). \]

Therefore Re f_\infty'(0)/f_\infty(0) > 0, and the inner geodesic normal to L_\infty(r) = \{z: Re log f_\infty(z) = n log r\} at 0 lies in \{Re z < 0\} (except for the point 0), in contradiction to our assumption. Theorem 2 follows because f_\infty(z) \to g(z) locally uniformly in G.

4. We shall apply Theorem 2 to obtain a result about the distortion under the conformal mapping of an annulus. It is a generalization of Theorem 6 in [5]. The closure of the region inside D that lies between two geodesies with common endpoint \zeta will be called a geodesic sector of vertex \zeta.

THEOREM 3. Let G be a doubly connected region in D, with \{|z| = 1\} as outer and E as inner boundary. Let w = g(z) be the function that maps G conformally onto \{ρ < |w| < 1\} such that g(1) = 1. Let S be the smallest geodesic sector of vertex 1 that contains E, and let T be the component of S \ E that contains 1. If R is the curve that w = g(z) maps onto the interval (ρ, 1) then R ⊆ T.
By the conformal mapping \( z^* = (1+z)/(1-z) \) of \( D \) onto \( \{ \text{Re} \, z^*>0 \} \) we see that Theorem 3 is equivalent with

**Theorem 3*. Let \( G^* \) be a doubly connected region in \( \{ \text{Re} \, z^*>0 \} \) with \( \{ \text{Re} \, z^*=0 \} \) and \( E^* \) as boundaries. Let \( w = g^*(z) \) map \( G^* \) conformally onto \( \{ \rho < |w| < 1 \} \) such that \( g^*(\infty) = 1 \). Let \( S^* \) be the smallest strip parallel to the real axis that contains \( E^* \), and let \( T^* \) be the component of \( S^* \setminus E^* \) that contains \( +\infty \). If \( R^* \) is the curve that \( w = g^*(z) \) maps onto the interval \( (\rho, 1) \) then \( R^* \subset T^* \).

**Proof.** Theorem 2 shows that all tangents to \( R^* \) certainly intersect \( S^* \). If \( S^* = \{ a \leq \text{Im} \, z^* \leq b \} \) it follows that

\[
\limsup_{z^* \in R^*} \text{Im} \, z^* \leq b.
\]

Also, all accumulation points of the left end of \( R^* \) lie on \( E^* \), hence in \( \{ \text{Im} \, z^* \leq b \} \). Suppose \( \max_{z^* \in R^*} \text{Im} \, z^* > b \). Together with (15) this would imply that the maximum is assumed, say at \( z = c \). But \( c \in S^* \), and the tangent to \( R^* \) at \( c \) is parallel to \( S^* \) so that it would not intersect \( S^* \). Thus we have shown that \( \text{Im} \, z^* \leq b \), and also \( \text{Im} \, z^* \geq a \), for \( z^* \in S^* \). Hence \( R^* \subset S^* \). Since \( R^* \) is a curve and contains points with large real part, \( R^* \) has to lie in the component \( T^* \) of \( S^* \setminus E^* \) that contains the point \( +\infty \).

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**References**


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