ON THE HYPERBOLIC CAPACITY AND CONFORMAL MAPPING

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1. Let \( E \) be a compact set in \( D = \{ |z| < 1 \} \). Tsuji [6] has introduced the hyperbolic capacity of \( E \) which can be defined by

\[
\caph E = \lim_{n \to \infty} \max_{z_0, \ldots, z_n \in E} \prod_{\mu=0}^{n} \frac{|z_\mu - z_\nu|}{1 - \bar{z}_\nu z_\mu}^{1/n(n+1)}.
\]

Also,

\[
\min_{f} \max_{z \in E} |f(z)|^{1/n} \to \caph E
\]

as \( n \to \infty \) where the minimum is taken over all functions

\[
f(z) = \prod_{r=1}^{n} e^{i\alpha_r} (z - z_r)/(1 - \bar{z}_r z) \quad (\alpha_r \text{ real, } |z_r| < 1).
\]

We shall first obtain another formula for \( \caph E \). Leja [1] has proved an analogous formula for the capacity of a plane set.

**Lemma.** Let \( E \) be a compact set in \( D \). For each \( n = 1, 2, \ldots \) choose \( n+1 \) points \( z_0, \ldots, z_n \) in \( E \) such that

\[
A_n = \prod_{\mu=0}^{n} \frac{|z_\mu - z_\nu|}{|1 - \bar{z}_\nu z_\mu|}
\]

becomes maximal. Numerate these points so that

\[
A_n = \prod_{r=1}^{n} |z_0 - z_r|/|1 - \bar{z}_r z_0|
\]

\[
= \min_{\mu} \prod_{\nu \neq \mu} |z_\mu - z_\nu|/|1 - \bar{z}_\nu z_\mu|.
\]

If

\[
f_n(z) = \prod_{r=1}^{n} \frac{1 - \bar{z}_r z - z_r}{1 - z_r (1 - \bar{z}_r z)}
\]

then

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(6) \[ \max_{z \in E} |f_n(z)| = A_n, \]
and, as \( n \to \infty \)
\[ A_n^{1/n} \to \text{caph } E. \]

**Proof.** For \( |\xi_1| < 1, |\xi_2| < 1 \) we write \( [\xi_1, \xi_2] = |\xi_1 - \xi_2| \left/ |1 - \xi_1 \xi_2| \right. \).
Let \( z \in E \). Comparing the system \( z, z_1, \ldots, z_n \) of points in \( E \) with the maximal system \( z_0, z_1, \ldots, z_n \) we see that
\[
1 \cdot [z, z_1] \cdots [z, z_n] \quad 1 \cdot [z_0, z_1] \cdots [z_0, z_n] \\
[z_1, z] \cdot 1 \cdots [z_1, z_n] \leq [z_1, z_0] \cdot 1 \cdots [z_1, z_n] \\
\vdots \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \vdots \\
[z_n, z][z_n, z_1] \quad \cdots \quad 1 \quad [z_n, z_0][z_n, z_1] \quad \cdots \quad 1.
\]
Hence \( |f_n(z)| \leq A_n \), with equality for \( z = z_0 \), which proves (6). Since \( f_n \) has the form (3) it follows that \( \min \max_{z \in E} |f(z)| \leq A_n \). Therefore by (2)
\[ \liminf_{n \to \infty} A_n^{1/n} \geq \text{caph } E. \]

On the other hand, (4) implies
\[ A_{n+1}^n \leq \prod_{\mu=0}^n \prod_{\rho \mu} [z_\mu, z_\nu]. \]
Hence (1) shows that \( \limsup_{n \to \infty} A_n^{1/n} \leq \text{caph } E \), and the Lemma follows.

2. Let \( E \) be a compact set in \( D = \{ |z| < 1 \} \). Then \( D \setminus E \) is an open set of which exactly one component region \( G \) has \( \{ |z| = 1 \} \) as part of the boundary. I shall give an elementary proof of the following theorem.

**Theorem 1.** Let \( \rho = \text{caph } E > 0 \). If \( f_n(z) \) is defined by (5) then
\[ g(z) = \lim_{n \to \infty} f_n(z)^{1/n} \]
exists locally uniformly in \( H = G \cup \{ 1 \leq |z| < r \} \) for some \( r > 1 \), and \( g(z) \) is the smallest function satisfying
(a) \( g(z) \) is locally analytic, and of single-valued modulus in \( H \),
(b) \( |g(z)| = 1 \) for \( |z| = 1 \),
(c) \( 1 \geq |g(z)| \geq \rho \) for \( z \in G \),
that is, if \( h(z) \) also satisfies these three conditions then \( |g(z)| \leq |h(z)| \) for \( z \in G \).

\[
\text{This means that } g(z) \text{ is analytic on the universal covering surface of } H.
\]
Furthermore, $g(1) = 1$ and

\[ \int_0^{2\pi} d \arg g(e^{i\theta}) = 2\pi. \]

If $\xi$ is a boundary point of $G$ that lies on a continuum contained in $E$ then $|g(z)| \to \rho$ for $z \to \xi$, $z \in G$.

Finally, if $E$ is a continuum then $\rho > 0$, and $w = g(z)$ maps $G$ conformally and one-to-one onto $\{ \rho < |w| < 1 \}$.

REMARKS. Let

\[ \omega(z) = \log(\rho^{-1} |g(z)|) / \log \rho^{-1}. \]

Then Theorem 1 shows that $\omega(z)$ is the smallest function satisfying

(a') $\omega(z)$ is single-valued and harmonic in $H$,
(b') $\omega(z) = 1$ for $|z| = 1$,
(c') $1 \leq \omega(z) \leq 0$ for $z \in G$,

that is, if $\nu(z)$ also satisfies these three conditions then $\omega(z) \leq \nu(z)$.

If the boundary of $G$ consists of a finite number of nondegenerate continua then $\omega(z) = 0$ on the boundary points of $G$ that lie in $D$.

Hence $\omega(z)$ is the harmonic measure of $\{|z| = 1\}$ with respect to $G$. By (8),

\[ 1 / \log \rho^{-1} = \frac{1}{2\pi} \int_0^{2\pi} \frac{\partial}{\partial \theta} \omega(re^{i\theta}) \bigg|_{r=1} d\theta. \]

Of course, we could have started with the harmonic measure and then proved (9). But the method applied here is simpler and more constructive. It does not use set-functions, the solvability of the Dirichlet problem or the Riemann mapping theorem. The existence of a function that maps a doubly-connected region onto an annulus is established (see also [4]).

The following proof uses (with some simplifications) the method of extremal points developed by Leja [1; 2; 3].

PROOF. a. The Lagrange interpolation formula shows that

\[ \sum_{\mu=0}^n \left( \prod_{\nu \neq \mu} \frac{z - z_\nu}{z_\mu - z_\nu} \cdot \prod_{r=1}^n \frac{1 - \bar{z}_\mu z_r}{1 - \bar{z}_\mu z_r} \right) = \prod_{r=1}^n \left( 1 - \bar{z}_\mu z_r \right). \]

Hence

\[ \max_{\mu} \left( \prod_{\nu \neq \mu} \frac{|z - z_\nu|}{|z_\mu - z_\nu|} \cdot \prod_{r=1}^n \frac{|1 - \bar{z}_\mu z_r|}{|1 - \bar{z}_\mu z_r|} \right) \geq \frac{1}{n + 1}. \]

Let $q(z) = \min_{1 \leq \mu \leq n} \frac{|z - \xi_\mu|}{|z - \xi_1|} |z - \xi_2|$ (for $z \in G$). Since $E \subset \{|z| \leq a\}$ for some $a < 1$ it follows that
\[ \max_{\mu} \left( \prod_{\mu=1}^{n} \left| \frac{z - z_\mu}{1 - \bar{z}_\mu z} \right| \cdot \prod_{\mu \in \mu} \left| \frac{1 - \bar{z}_\mu z}{z_\mu - z} \right| \right) \geq \frac{(1 - a^2)q(z)}{2(n + 1)}, \]

and because of (4)

\[ |f_n(z)| \geq \frac{(1 - a^2)q(z)}{2(n + 1)}. \]

We put \( r = 2/(1 + a) > 1 \). Since \( |z - z_\mu| / |1 - \bar{z}_\mu z| \leq (r + a)/(1 - ar) < 4/(1 - a) \) for \( |z| \leq r \), (5) shows that

\[ |f_n(z)|^{1/n} < 4/(1 - a) \quad (|z| \leq r). \]

b. Let \( H = G \cup \{ 1 \leq |z| \leq r \} \) and \( g_n(z) = f_n(z)^{1/n} \). The functions \( g_n(z) \) are locally analytic in \( H \), and \( |g_n(z)| \) is single-valued. By (11) and Montel’s theorem we can find a sequence \( n_k \) such that \( g_{n_k}(z) \) converges locally uniformly in \( H \). Let \( g(z) \) be the limit function. Since by the Lemma \( A_n^{1/n} \to \rho \), inequality (10) implies \( |g(z)| \geq \rho \). Also \( |g(z)| = 1 \) for \( |z| = 1 \) so that \( g(z) \) satisfies (a), (b) and (c).

Let \( h(z) \) be any function satisfying these three conditions, and let \( z^* \) be a point in \( G \). Given \( \epsilon > 0 \) we choose a fixed \( k \) so large that \( |g_{n_k}(z^*)| > e^{-\epsilon} |g(z^*)| \). Since \( \rho > 0 \) we can take \( k \) so that also \( A_n^{1/n} < \rho e^\epsilon \). Then it follows from (6) that \( |g_{n_k}(z)| \leq \rho e^\epsilon \) for \( z \in E \). We choose analytic curves in \( G \) so near to \( E \) that their union \( C \) separates \( E \) from \( z^* \) and from \( \{ |z| = 1 \} \), and that \( |g_{n_k}(z)| \leq \rho e^{2\epsilon} \) for \( z \in C \). Because \( |h(z)| \geq \rho \) for \( z \in G \),

\[ |g_{n_k}(z)| / |h(z)| \leq \rho e^{2\epsilon} / \rho = e^{2\epsilon} \]

for \( z \in C \). Since the left side is \( = 1 \) for \( |z| = 1 \) it follows from the maximum principle that the inequality holds also for \( z = z^* \). Hence

\[ |g(z^*)| < e^\epsilon |g_{n_k}(z^*)| \leq e^{2\epsilon} |h(z^*)| \]

for every \( \epsilon > 0 \) and therefore \( |g(z^*)| \leq |h(z^*)| \).

Since \( f_n(1) = 1 \) we obtain \( g(1) = 1 \), and by the argument principle

\[ \int_0^{2\pi} d \arg g_n(e^{i\theta}) = \frac{1}{n} \int_0^{2\pi} d \arg f_n(e^{i\theta}) = 2\pi \]

from which (8) follows.

If \( g_n(z) \) did not converge there would be a limit function \( h(z) \neq g(z) \) for some other subsequence of \( g_n \) as Montel’s theorem shows. From what we have proved it follows that \( |h(z)| \geq |g(z)| \). Reversing the roles of \( g \) and \( h \) we also get \( |g(z)| \geq |h(z)| \). Hence \( |h(z)| = |g(z)| \), and \( g(1) = h(1) = 1 \) implies \( g(z) \equiv h(z) \). Therefore \( g_n(z) \to g(z) \) as \( n \to \infty \).

c. We assume now that \( E \) is a continuum. We do not know yet that
\( \rho > 0 \). If \( \rho = 0 \) then \( \lim_{n \to \infty} g_n(z) \) might not exist. In this case let \( g(z) \) be the limit function in \( H \) for some convergent subsequence of \( g_n \) which exists by (11). The region \( H \) is doubly connected, and every simply closed curve in \( H \) is homotopic either to a point or the unit circle. Therefore the functions \( g(z) \) and \( g_n(z) \) are single-valued because of (8) and (12). Let \( c \) be any point with \( \rho < |c| < 1 \). It follows from (12) that \( w = g_n(z) \) maps \( \{|z| = 1\} \) one-to-one onto \( \{|w| = 1\} \). Hence

\[
\left(13\right) \quad \frac{1}{2\pi i} \int_{|z|=1} \frac{g_n'(z)}{g_n(z) - c} \, dz = \frac{1}{2\pi i} \int_{|w|=1} \frac{1}{w - c} \, dw = 1.
\]

We choose analytic curves \( C_m \) \((m = 1, 2, \ldots)\) enclosing \( E \) so that the regions between \( C_m \) and \( \{|z| = 1\} \) approach \( G \). By the Lemma we can choose them so that \( |g_n(z)| < |c| \) on each \( C_m \) for sufficiently large \( n \). Then

\[
\left(14\right) \quad \frac{1}{2\pi i} \int_{C_m} \frac{g_n'(z)}{g_n(z) - c} \, ds = 0.
\]

Making \( n \to \infty \) we obtain from (13) and (14) that

\[
\frac{1}{2\pi i} \int_{|z|=1} \frac{g'(z)}{g(z) - c} \, dz - \frac{1}{2\pi i} \int_{C_m} \frac{g'(z)}{g(z) - c} \, ds = 1
\]

for all \( m \). Hence \( g(z) \) assumes the value \( c \) exactly once in \( G \). Therefore \( w = g(z) \) maps \( G \) one-to-one onto \( \{\rho < |w| < 1\} \).

Suppose that \( \rho = 0 \) were true. Then the inverse function \( \psi(w) \) of \( w = g(z) \) would be analytic and univalent in \( \{0 < |w| \leq 1\} \). Since \( |\psi(w)| < 1 \) it would follow that \( \psi(w) \) is bounded and univalent in \( \{|w| \leq 1\} \). Since \( |\psi(w)| = 1 \) for \( |w| = 1 \) this would imply that \( \psi(w) \) is a linear function and therefore \( E \) a point.

d. Let again \( E \) be arbitrary and let \( \xi \) be a boundary point lying on a continuum \( B \) that is contained in \( E \). Let \( g_0(z) \) be the function that maps the doubly-connected region between \( B \) and \( \{|z| = 1\} \) onto \( \{|\rho_0 < |w| < 1\} \). Let \( \lambda \) be such that \( \rho_0^\lambda = \rho \). Then \( h(z) = g_0(z)^\lambda \) satisfies the conditions (a), (b) and (c) of Theorem 1. Hence, as we have already proved, \( \rho \leq |g(z)| \leq |h(z)| = |g_0(z)|^\lambda \). Since for simple topological reasons \( |g_0(z)| \to \rho_0 \) as \( z \to \xi, \xi \in G \) it follows that \( |g(z)| \to \rho \).

3. We shall now prove an analogue to a result by Walsh [7] about the ordinary Green's function. We introduce the hyperbolic metric in the unit disk \( D \). A circle perpendicular to \( \{|z| = 1\} \) will be called a geodesic.

**Theorem 2.** Let \( E \) be a compact set in \( D \) with \( \caph E > 0 \), and let
$g(z)$ be the function defined in Theorem 1. Let $L(r) = \{z: |g(z)| = r\}$ (caph $E < r < 1$). Then at every point of $L(r)$ the inner geodesic normal to $L(r)$ intersects the hyperbolically convex hull $K$ of $E$.

**Remarks.** The hyperbolically convex hull of $E$ is defined as the smallest closed set $K \supseteq E$ that is convex in the hyperbolic metric in the sense that together with any two points also the geodesic segment between these two points belongs to $K$. The set $L(r)$ is the union of a finite number of closed analytic curves which may have multiple points though. At the multiple points $g'(z)$ vanishes. It is easy to see that all multiple points lie in $K$ (see [8, p. 157]).

**Proof.** With the notations of Theorem 1 let $L_n(r) = \{z: |f_n(z)| = r^n\}$. We first prove that the inner geodesic normal to $L_n(r)$ at any $\xi \in L_n(r)$ intersects $K$. Suppose this were false. Then $\xi \in K$. By a conformal mapping of the unit disk onto itself we can make $\xi = 0$. Then the inner geodesic normal becomes a straight ray and is separated from $K$ by a line. We may thus assume that $K \subseteq \{\text{Re } z < 0\}$ and that the inner normal lies in $\{\text{Re } z \geq 0\}$. Writing $z_r = x_r + iy$, we have $x_r < 0$. Hence

$$
\frac{d}{dz} \log f_n(z) \bigg|_{z=0} = \sum_{r=1}^{n} \frac{1 - |z_r|^2}{(z - z_r)(1 - \bar{z}_r z)} \bigg|_{z=0} = \sum_{r=1}^{n} \frac{1 - |z_r|^2}{|z_r|^2} (-x_r + i y_r).
$$

Therefore $\text{Re } f_n'(0)/f_n(0) > 0$, and the inner geodesic normal to $L_n(r) = \{z: \text{Re } \log f_n(z) = n \log r\}$ at $0$ lies in $\{\text{Re } z < 0\}$ (except for the point $0$), in contradiction to our assumption. Theorem 2 follows because $f_n(z)^{1/n} \to g(z)$ locally uniformly in $G$.

4. We shall apply Theorem 2 to obtain a result about the distortion under the conformal mapping of an annulus. It is a generalization of Theorem 6 in [5]. The closure of the region inside $D$ that lies between two geodesics with common endpoint $\xi$ will be called a geodesic sector of vertex $\xi$.

**Theorem 3.** Let $G$ be a doubly connected region in $D$, with $\{|z| = 1\}$ as outer and $E$ as inner boundary. Let $w = g(z)$ be the function that maps $G$ conformally onto $\{\rho < |w| < 1\}$ such that $g(1) = 1$. Let $S$ be the smallest geodesic sector of vertex $1$ that contains $E$, and let $T$ be the component of $S \setminus E$ that contains $1$. If $R$ is the curve that $w = g(z)$ maps onto the interval $(\rho, 1)$ then $R \subset T$. 

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By the conformal mapping $z^* = (1+z)/(1-z)$ of $D$ onto $\{\Re z^* > 0\}$ we see that Theorem 3 is equivalent with

**Theorem 3*. Let $G^*$ be a doubly connected region in $\{\Re z^* > 0\}$ with $\{\Re z^* = 0\}$ and $E^*$ as boundaries. Let $w = g^*(z)$ map $G^*$ conformally onto $\{\rho < |w| < 1\}$ such that $g^*(\pm \infty) = 1$. Let $S^*$ be the smallest strip parallel to the real axis that contains $E^*$, and let $T^*$ be the component of $S^* \setminus E^*$ that contains $\pm \infty$. If $R^*$ is the curve that $w = g^*(z)$ maps onto the interval $(\rho, 1)$ then $R^* \subset T^*$.

**Proof.** Theorem 2 shows that all tangents to $R^*$ certainly intersect $S^*$. If $S^* = \{a \leq \Im z^* \leq b\}$ it follows that

$$(15) \limsup_{z^* \in R^*} \Im z^* \leq b.$$ 

Also, all accumulation points of the left end of $R^*$ lie on $E^*$, hence in $\{\Im z^* \leq b\}$. Suppose $\max_{z^* \in R^*} \Im z^* > b$. Together with (15) this would imply that the maximum is assumed, say at $z = c$. But $c \in S^*$, and the tangent to $R^*$ at $c$ is parallel to $S^*$ so that it would not intersect $S^*$. Thus we have shown that $\Im z^* \leq b$, and also $\Im z^* \geq a$, for $z^* \in S^*$. Hence $R^* \subset S^*$. Since $R^*$ is a curve and contains points with large real part, $R^*$ has to lie in the component $T^*$ of $S^* \setminus E^*$ that contains the point $\pm \infty$.

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**References**


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