SHORTER NOTES

The purpose of this department is to publish very short papers of an unusually elegant and polished character, for which there is normally no other outlet.

A SIMPLE EXAMPLE OF A TRANSCENDENTAL ENTIRE FUNCTION THAT TOGETHER WITH ALL ITS DERIVATIVES ASSUMES ALGEBRAIC VALUES AT ALL ALGEBRAIC POINTS

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Let \( \{ z_i \} = \{ z_1, z_2, z_3, \ldots \} \) be an enumeration of all algebraic numbers [1]. Construct a sequence \( \{ z_j \} = \{ z_1, z_2, z_3, \ldots \} \) so that all of the algebraic numbers appear an infinite number of times in \( \{ z_j \} \). Then, for algebraic numbers \( a_n \) with \( 0 < |a_n| < (n! \cdot \prod_{j=1}^{n} (1 + |z_j|))^{-1} \), the function \( f(z) = \sum_{n=0}^{\infty} a_n \cdot \prod_{j=1}^{n} (z-z_j) \) is an entire function having the said property. Since \( |z-z_j| \leq 1 + |z_j| \) for \( |z| \leq 1 \) and \( |z-z_j| \leq |z| \cdot (1 + |z_j/z|) \) for \( |z| > 1 \), the series for \( f(z) \) converges absolutely and uniformly in \( |z| \leq R < \infty \) and \( |f(z)| \leq \max \{ e, e^{\left|z\right|} \} \). Since \( f^{(m)}(z_j) \) is a polynomial of \( z_j \) with algebraic coefficients \( a_n \) and \( \{ z_j \} \) contains all algebraic numbers infinitely many times, \( f^{(m)}(z) \) must be an algebraic number for any algebraic number \( z \).

If we ask the general question: For what sets, \( S \), of complex numbers do there exist transcendental entire functions which, together with all their derivatives, map \( S \) into \( S' \), we see immediately that the above construction can be applied to any dense denumerable set, or to any denumerable ring which has 0 as a limit point, such as the ring of rationals. A similar method can be applied to discrete infinite rings such as the ring of integers. The question for nondenumerable nonclosed rings \( S \) remains open.

REFERENCES


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