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NOTE ON A NONLINEAR VOLTERRA EQUATION

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1. Introduction. We investigate the solutions of

\[ x'(t) = - \int_0^t a(t - \tau)g(x(\tau))d\tau \]  

as \( t \to \infty \), where \( a(t) \) is completely monotonic on \( 0 \leq t < \infty \) and where \( g(x) \) is a (nonlinear) spring. Under this hypothesis, (1.1) was shown in [2] to be relevant to certain physical applications and results were obtained there for the linear case \( g(x) = x \). (If \( a(t) = a(0) \), then (1.1) reduces to the nonlinear oscillator \( x'' + a(0)g(x) = 0 \).) Equation (1.1) was studied in [1] under less hypothesis on \( a(t) \). However, while the result is weaker than that of [1], the present approach draws together such different notions of positivity as Liapounov functions, completely monotonic functions, and kernels of positive type. It also provides a new Liapounov function for (1.1). Specifically, we prove the

**Theorem.** Let \( a(t) \) and \( g(x) \) satisfy

\[ a(t) \in C[0, \infty), (-1)^ka^{(k)}(t) \geq 0 \quad (0 < t < \infty ; k = 0, 1, 2, \ldots), \]

\[ g(x) \in C(-\infty, \infty), \quad xg(x) > 0 \quad (x \neq 0), \quad G(x) = \int_0^x g(\xi)d\xi \to \infty \]  

\[ (|x| \to \infty). \]  

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If \( a(t) \neq a(0) \) and if \( u(t) \) is any solution of (1.1) which exists on \( 0 \leq t < \infty \), then

\[
\lim_{t \to \infty} u^{(j)}(t) = 0 \quad (j = 0, 1, 2).
\]

In [1] only \( k = 0, 1, 2, 3 \) is required in the analogue of (1.2), rather than complete monotonicity. The Liapounov function used there was

\[
E(t) = G(u(t)) + \frac{1}{2} a(t) \left[ \int_0^t g(u(\tau)) d\tau \right]^2
\]

In (1.5) and the sequel \( u(t) \) is the solution of (1.1) on \( 0 \leq t < \infty \) being considered. Remarks concerning existence, uniqueness, as well as background information and references, may be found in [1].

2. Positivity. In this section we motivate the hypothesis (1.2) and also obtain the Liapounov function. Suppose that the origin of the problem is such that \( a(t) \neq 0, a(t) \in C[0, \infty), a'(t) \in L_1(0, T) \) for each \( 0 < r < \infty \), and (1.3) are all required. Then clearly

\[
V(t) = G(u(t)) + \frac{1}{2} \int_0^t a(t + \tau) g(u(t - \tau)) g(u(t - s)) d\tau ds
\]

is nonnegative if the second term is. (From (1.1) and (1.3), \( V(t) \) may be interpreted as the sum of a potential and kinetic energy.)

The second (or kinetic) term of \( V(t) \) will be nonnegative if it is assumed that \( a(\tau + s) \) is a kernel of positive type [3, p. 270] on the square \( 0 < \tau, s < \tau \) for each \( 0 < t < \infty \). By a theorem of Boas and Widder [3, pp. 273–275], this will be the case if and only if

\[
a(t) = \int_{-\infty}^{\infty} \exp[-\xi t] d\alpha(\xi) \quad (0 < t < \infty),
\]

where \( \alpha(\xi) \) is nondecreasing on \( -\infty < \xi < \infty \) (and may be assumed normalized: \( \alpha(0) = 0, \alpha(\xi) = \frac{1}{2} [\alpha(\xi^+) + \alpha(\xi^-)] \)).

Differentiating (2.1) yields

\[
V'(t) = \int_0^t \int_0^t a'(t + \tau) g(u(t - \tau)) g(u(t - s)) d\tau ds,
\]

where we have used the above assumptions, (1.1), the absolutely continuity of \( a(t) \) on \( 0 < t \leq T < \infty \) implied by (2.2), and an obvious change of variables. If \( V(t) \) is to serve as a Liapounov function for
(1.1), then \( V'(t) \) must be nonpositive. By (2.3) the latter will be assured if \(-a'(t+s)\) is a kernel of positive type on the square \( 0 < r, s < t \) for each \( 0 < t < \infty \). However, this is compatible with (2.2) if and only if \( \alpha(-\infty) = \alpha(0-) \). Thus, also using \( a(t) \in C[0, \infty) \), one has

\[
(2.4) \quad a(t) = \int_0^\infty \exp[-\xi t]da(\xi) \quad (0 \leq t < \infty),
\]

where \( \alpha(\infty) < \infty \). By a theorem of S. Bernstein [3, p. 160], (1.2) and (2.4) are equivalent.

Having motivated the hypothesis (1.2) and obtained a Liapounov function \( V(t) \), it is interesting to compare the two Liapounov functions (1.5) and (2.1). It is obvious that if \( a(t) = a(0) \), then \( E(t) = V(t) \ (0 \leq t < \infty) \). We now prove a strong form of the converse statement. In particular, if \( u(0) \neq 0 \) and if \( E(t) \equiv V(t) \ (0 \leq t \leq t_0) \) for some \( 0 < t_0 < \infty \), then \( a(t) \equiv a(0) \ (0 \leq t < \infty) \). For this one need only assume that \( a(t) \in C[0, \infty) \), \((-1)^ka^{(k)}(t) \geq 0 \) \((0 < t < \infty; k = 0, 1, 2)\), \( g(x) \in C(-\infty, \infty) \), and \( g(x) = 0 \) implies \( x = 0 \). Direct calculations show that (1.5) and (2.1) may also be written as

\[
E(t) = G(u(t)) + \int_0^t g(u(\tau)) \left\{ \int_\tau^t a(t - \tau)g(u(s))ds \right\} d\tau,
\]

\[
V(t) = G(u(t)) + \int_0^t g(u(\tau)) \left\{ \int_\tau^t a(2t - \tau - s)g(u(s))ds \right\} d\tau.
\]

Hence

\[
\int_0^t g(u(\tau)) \left\{ \int_\tau^t [a(t - \tau) - a(2t - \tau - s)]g(u(s))ds \right\} d\tau = 0 \quad (0 \leq t \leq t_0).
\]

(2.5)

From the hypothesis on \( a(t) \) one has \( a(t-\tau) - a(2t-\tau-s) \geq 0 \) for \( 0 \leq \tau, s \leq t \). As \( u(0) \neq 0 \), there exists by continuity and the hypothesis on \( g(x) \) a \( 0 < t_1 \leq t_0 \) such that \( g(u(t)) \neq 0 \) for \( 0 \leq t \leq t_1 \). It is now clear from (2.5) that \( a(t-\tau) - a(2t-\tau-s) \) \((0 \leq \tau \leq t \leq t_1)\). The latter together with \( a(t) \in C[0, \infty) \) easily implies \( a(t) \equiv a(0) \) \((0 \leq t \leq 2t_1)\). Hence \( a'(t) \equiv 0 \) \((0 \leq t \leq 2t_1)\). However, as \(-a'(t), a''(t) \geq 0 \) it follows that \( a'(t) \equiv 0 \) \((0 \leq t < \infty)\) so that \( a(t) \equiv a(0) \) \((0 \leq t < \infty)\), which proves the assertion.

3. Proof of the Theorem. Let \( a(t), V(t) \) be given by (2.4), (2.1) respectively. Define
(3.1) \[ \Gamma(\xi, t) = \int_0^t \exp\left[ -\xi(t - \tau) \right] g(u(\tau)) \, d\tau \quad (0 \leq \xi, t < \infty). \]

Then

(3.2) \[ \frac{\partial \Gamma}{\partial t} (\xi, t) = g(u(t)) - \xi \Gamma(\xi, t) \quad (0 \leq \xi, t < \infty). \]

Clearly \( \Gamma(\xi, t), \Gamma_t(\xi, t) \) are bounded functions of \( \xi \) on \( 0 \leq \xi < \infty \) for each fixed \( t \) in \( 0 \leq t < \infty \). Hence, by Fubini's theorem, one may write (1.1) (with \( u(t) \) replacing \( x(t) \)), (2.1) as

(3.3) \[ u'(t) = -\int_0^\infty \Gamma(\xi, t) \, d\alpha(\xi) \quad (0 \leq t < \infty), \]

(3.4) \[ V(t) = G(u(t)) + \frac{1}{2} \int_0^\infty \Gamma^2(\xi, t) \, d\alpha(\xi) \geq 0 \quad (0 \leq t < \infty) \]

respectively and, moreover,

(3.5) \[ V'(t) = -\int_0^\infty \xi \Gamma^2(\xi, t) \, d\alpha(\xi) \leq 0 \quad (0 \leq t < \infty). \]

From (3.4), (3.5) one has

\[ G(u(t)) \leq V(t) \leq V(0) = G(u_0) \quad (0 \leq t < \infty), \]

where \( u_0 = u(0) \). It follows from (1.3) that

(3.6) \[ \left| u(t) \right| \leq K < \infty \quad (0 \leq t < \infty). \]

In (3.6) and subsequent formulas \( K = K(u_0) < \infty \), where \( K \) may vary from formula to formula, and \( K(u_0) \rightarrow 0 \) as \( u_0 \rightarrow 0 \). Thus

(3.7) \[ \left| \Gamma(\xi, t) \right| \leq Kt, \quad \left| \xi \Gamma(\xi, t) \right| \leq K, \quad \left| \Gamma_t(\xi, t) \right| \leq K \quad (0 \leq \xi, t < \infty). \]

Differentiating (3.3) (using Fubini's theorem) yields

(3.8) \[ u''(t) = -g(u(t)) \alpha(\infty) + \int_0^\infty \xi \Gamma(\xi, t) \, d\alpha(\xi) \quad (0 \leq t < \infty), \]

which together with (1.3), (3.6), and (3.7) implies

(3.9) \[ \left| u''(t) \right| \leq K \quad (0 \leq t < \infty). \]

By (3.6), (3.9), and the mean value theorem one has

(3.10) \[ \left| u'(t) \right| \leq K \quad (0 \leq t < \infty). \]

From (3.2), (3.5) there results
\[
V''(t) = -2g(u(t)) \int_0^\infty \xi \Gamma(\xi, t) d\alpha(\xi) + 2 \int_0^\infty \xi^2 \Gamma^2(\xi, t) d\alpha(\xi)
\]

so that by (1.3), (3.6), (3.7) one has \(|V''(t)| \leq K\) on \(0 \leq t < \infty\). The latter together with (3.4), (3.5), and the mean value theorem implies (see Lemma 1 of \([1]\))

\[
(3.11) \quad \lim_{t \to \infty} V'(t) = -\lim_{t \to \infty} \int \xi \Gamma^2(\xi, t) d\alpha(\xi) = 0.
\]

We assert that there exists a \(\xi_1 > 0\) such that \(\Gamma(\xi_1, t) \to 0\) as \(t \to \infty\). Suppose not and let \(0 < \xi_0 < \infty\). Then there exist a \(\lambda = \lambda(\xi_0) > 0\) and a sequence \(\{t_n = t_n(\xi_0)\}\), where \(t_n \to \infty\) as \(n \to \infty\), such that \(|\Gamma(\xi_0, t_n)| \geq \lambda\). From (3.1) one has

\[
(3.12) \quad \frac{\partial \Gamma}{\partial \xi}(\xi, t) = -\int_0^t \exp[-\xi(t - \tau)](t - \tau) g(u(\tau)) d\tau \quad (0 \leq \xi, t < \infty).
\]

Let \(\delta = \min(\xi_0 / 2, \lambda \xi_0^2 / 8K_1), I_{t_0} = \{\xi \mid \xi - \xi_0 \leq \delta\}\), where \(|g(u(t))| \leq K_1\) on \(0 \leq t < \infty\). Then by the mean value theorem and (3.12) one obtains \(|\Gamma(\xi, t_n) - \Gamma(\xi_0, t_n)| \leq \lambda / 2\) \((\xi \in I_{t_0})\) so that, as \(|\Gamma(\xi_0, t_n)| \geq \lambda, |\Gamma(\xi, t_n)| \geq \lambda / 2\) \((\xi \in I_{t_0})\). Hence

\[
\int_0^\infty \xi \Gamma^2(\xi, t_n) d\alpha(\xi) \geq \int_{I_{t_0}} \xi \Gamma^2(\xi, t_n) d\alpha(\xi)
\]

\[
\geq \lambda \xi_0^2 \int_{I_{t_0}} \left[ a(\xi_0 + \delta) - a(\xi_0 - \delta) \right] d\alpha(\xi)
\]

\[
\geq \lambda \xi_0^2 \left[ a(\xi_0 + \delta) - a(\xi_0 - \delta) \right] \geq 0,
\]

which with (3.11) yields \(a(\xi_0 + \delta) = a(\xi_0 - \delta)\). As this is true for each \(0 < \xi_0 < \infty\), it follows that \(a(0+) = a(\infty)\) which contradicts \(a(0) \neq a(0)\). Thus, there exists a \(\xi_1 > 0\) with the asserted property.

Let \(f(t) = \exp[-\xi t], \ p(t) = g(u(t))\) for \(0 \leq t < \infty\) and \(f(t) = p(t) = 0\) for \(- \infty < t < 0\). From the preceding paragraph one has

\[
(3.13) \quad \lim_{t \to \infty} \Gamma(\xi_1, t) = \lim_{t \to \infty} \int_{-\infty}^\infty f(t - r) p(\tau) d\tau = 0.
\]

By applying Pitt’s form of Wiener’s tauberian theorem \([3, p. 211]\) to (3.13), we now show that \(p(t) \to 0\) as \(t \to \infty\). (A longer elementary argument could also be used here.) Clearly, \(f \in L_1(- \infty, \infty)\) and its Fourier transform \(\hat{f}(s) = (2\pi)^{-1/2}(\xi_1 + is)^{-1} \neq 0\) for \(- \infty < s < \infty\). As \(|p(t)| \leq K\) \((- \infty < t < \infty\), there remains only to show that \(p(t)\) is a slowly decreasing function in \((- \infty, \infty)\). For this it suffices to show
that \( p(t_i) - p(s_i) \to 0 \) as \( i \to \infty \) if \( \{t_i\}, \{s_i\} \) are any sequences satisfying \( t_i > s_i > 0 \) and \( s_i \to \infty, t_i - s_i \to 0 \) as \( i \to \infty \). However, from (1.3), (3.6), (3.10), the mean value theorem, and uniform continuity, it is clear that \( g(u(t_i)) - g(u(s_i)) \to 0 \) as \( i \to \infty \) for such sequences. Thus, \( p(t) \to 0 \) as \( t \to \infty \), which together with (1.3) and (3.6) yields \( u(t) \to 0 \) as \( t \to \infty \) (i.e., (1.4, \( j = 0 \))).

From (1.4, \( j = 0 \)) and (3.9) one has (1.4, \( j = 1 \)) by the mean value theorem. From (1.4, \( j = 0 \)) and (3.1) it is an elementary exercise to show that \( \xi \Gamma(\xi, t) \to 0 \) as \( t \to \infty \) uniformly with respect to \( \xi \) on \( 0 \leq \xi < \infty \). This together with (1.4, \( j = 0 \)), (1.3), and (3.8) implies (1.4, \( j = 2 \)), which completes the proof.

**References**


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