SOME BOUNDARY VALUE PROBLEMS FOR LINEAR DIFFERENTIAL SYSTEMS

W. J. COLES

1. Introduction. Let $A(t)$ and $f(t)$ be $n \times n$ and $n \times 1$ matrices, respectively, continuous on an interval $[a, b]$. In [1], J. B. Garner and L. P. Burton consider the boundary value problems

\begin{align*}
(1) \quad y' &= Ay + f, \quad y_i(a) = \beta_i \quad (1 \leq i < n), \quad y_n(b) = \beta_n \\
(2) \quad y' &= Ay + f, \quad y_i(a) = \beta_i, \quad y_i(c) = \beta_i \quad (1 < i < n, a < c < b), \quad y_n(b) = \beta_n,
\end{align*}

and prove:

**Theorem A.** If, for each $i$ and $j$ ($1 \leq i, j \leq n$, $i \neq j$), $a_{ij}a_{in}a_{jn} > 0$ and $a_{in}a_{nj} > 0$ on $[a, b]$, the problem (1) has a unique solution;

**Theorem B.** Under certain conditions on $A(t)$, too lengthy to give here, the problem (2) has a unique solution.

The authors note that Theorem A has a dual in which the roles of $a$ and $b$ are interchanged, provided that $a_{ij}a_{in}a_{jn} < 0$ is assumed.

The purpose here is to obtain theorems corresponding to Theorem A and its dual, with considerably less restriction on $A(t)$, and to use these results to obtain as a direct consequence a theorem corresponding to Theorem B.

2. The two-point problem. As usual, we rephrase the problem in terms of the homogeneous system. Let $N$ be fixed ($1 \leq N \leq n$); let $Q = (\delta_{ij} \delta_{in})$, and let $P = E - Q$ ($E$ being the $n \times n$ identity); let $\beta = \text{col}(\beta_i)$. Let $z(t)$ be a solution of $y' = Ay + f$ which does not satisfy

\begin{equation}
(3) \quad y' = Ay + f, \quad Py(a) + Qy(b) = \beta.
\end{equation}

If $X$ is any nonsingular solution of $X' = AX$, the general solution of $y' = Ay + f$ can be written in the form $Xc + z$, and our boundary condition reduces to

$$[PX(a) + QX(b)] \cdot c = \beta - Pz(a) - Qz(b) \neq 0.$$ 

Thus (3) has a unique solution if and only if the equation $X' = AX$
has a nonsingular solution for which \( PX(a) + QX(b) \) is nonsingular. We may assume that \( X(a) = E \); our condition is then that \( x_N(b) \neq 0 \) if \( x' = Ax \) and \( x_i(a) = \delta_{ii} \) (1 \( \leq i \leq n \)).

For convenience we list the following conditions and definitions.

\((0)\) \( a_{ii}(t) \equiv 0 \) (1 \( \leq i \leq n \)).

\((I)\) For \( N \) fixed (1 \( \leq N \leq n \)), there exist \( K \neq N \) (1 \( \leq K \leq n \)) and \( m_K \) (1 \( \leq m_K \leq n - 1 \)) such that no product \( a_{K1}(t_0)a_{j1j_2}(t_1) \cdots a_{jmN}(t_m) \), with at most \( m_K + 1 \) factors, changes sign on \( a \leq t_i \leq b \) (0 \( \leq i \leq m \)), each such product has the same sign, and one such product, with at most \( m_K \) factors, is nonzero at \( t = a \). Let \( s_{KN} \) be 1 or -1, according as this last product is positive or negative at \( t = a \); let \( s_{NN} = 1 \).

\((II)\) For \( N \) fixed, (I) holds for each \( K \neq N \); \( s_{KN}s_{NN}(t) \equiv 0 \); the \( m_K \)'s may be taken equal.

\((III)\) \( x' = Ax \) and \( x_i(a) = \delta_{ii} \) (1 \( \leq i \leq n \)).

**Lemma 1.** If \( x' = Ax \) and (0) holds, and if, for a fixed \( j_i \), \( \sigma_i \) is the set of integers including 1, \( \cdots \), \( n \), but excluding \( j_i \), then

\( (4i) \quad x_{i_0}(t_0) = x_{i_0}(a) + \sum_{j_1 \in \sigma_0} \int_a^{t_0} a_{j_1i_1}(t_1)x_{j_1}(t_1)dt_1; \)

\( (4ii) \quad x_{i_0}(t_0) = x_{i_0}(a) + \sum_{k=1}^{m} \sum_{j_1 \in \sigma_0} \cdots \sum_{j_{k+N-1} \in \sigma_{k-1}} \int_a^{t_0} dt_1 \cdots \)

\( \cdot \int_a^{t_k-1} \int_a^{t_{k-1}-1} \cdots \int_a^{t_{k-1}-1} \cdots \sum_{j_{k+N-1} \in \sigma_{k-1}} \int_a^{t_0} dt_1 \cdots \)

\( \cdot \int_a^{t_m-1} \int_a^{t_{m-1}-1} \cdots \sum_{j_{m+1} \in \sigma_m} \int_a^{t_0} \cdots \int_a^{t_{m-1}} \cdots \sum_{j_{m+1} \in \sigma_m} \int_a^{t_0} \cdots \int_a^{t_{m}} \cdots \int_a^{t_{m}} \cdots \int_a^{t_{m}} a_{j_{m+1,i_{m+1}}(t_{m+1})} dt_{m+1}, \quad m \geq 1. \)

**Proof.** Integration of \( x_{i_0}' \) gives (4i) and, in fact, a similar expression for each \( x_{i_j} \). Substituting these expressions into the right-hand side of (4i), and continuing the process, gives (4ii).

**Lemma 2.** If (0), (I), and (III) hold, there is a \( \delta > 0 \) such that \( s_{KN}x_K(t) > 0 \) on \( (a, a+\delta) \).

**Proof.** In (4), let \( j_0 = K \) and \( j_{m+1} = N \). By (III), the first term in each of (4i) and (4ii) is zero. Since all possible products with \( m + 1 \) factors and of the type in (I) occur in the last set of terms in (4i) or (4ii), proper choice of \( m \) will cause to appear a product involving \( x_N \) which is nonzero at \( a \). This term, and indeed all terms involving \( x_N \),
have the sign of $s_{KN}$. All other terms in the last set are zero at $a$; all nonzero terms in the second set of terms in (4ii) involve $x_N(a)$, and so have the sign of $s_{KN}$. Hence, for small positive $\delta$, $s_{KN}x_K(t) > 0$ on $(a, a + \delta)$.

**Corollary 1.** If (0), (II) and (III) hold, there is a $\delta > 0$ such that

$$s_{IN}x_i(t) > 0 \quad (1 \leq i \leq n)$$

on $(a, a + \delta)$.

**Lemma 3.** If (0), (II) and (III) hold, then (5) holds on $(a, b]$.

**Proof.** By Corollary 1, there is a $\delta$ such that $0 < \delta < b - a$ and for which (5) holds on $(a, a + \delta)$. Let $a < c < a + \delta$, and let $P(t) = x_1(t) \cdots x_n(t)$. Then

$$P(t) = P(c) \exp \int_c^t \sum_{k,j=1; k \neq j}^n a_{kj}x_j/x_k dt$$

on $(c, a + \delta)$. Now, for each $k$ and $j$ such that $k \neq j$, we have $a_{kj}x_j/x_k \geq 0$. Indeed, if $k = N$ or $j = N$ this follows from the condition $s_{IN}x_N(t) \geq 0$ in (II) and from the conclusion of Lemma 2. If $j \neq N$ and $k \neq N$, let $m$ be the common value of the $m_i$'s in (II). There is a product $P_{jN}$ of the form in (I) (with $K = j$), with at most $m$ factors, such that $s_{jN}P_{jN}(t) \geq 0$. Indeed, if $k = N$ or $j = N$ this follows from the conclusion of Lemma 2. If $j \neq N$ and $k \neq N$, let $m$ be the common value of the $m_i$'s in (II). There is a product $P_{jN}$ of the form in (I) (with $K = j$), with at most $m$ factors, such that $s_{jN}P_{jN}(t) \geq 0$. Hence we can write $a_{kj}x_j/x_k = [a_{kj}P_{jN}(a)/x_k] \cdot [x_j/P_{jN}(a)]$, with each factor non-negative. Thus the statement is verified. From this, $|P(t)| \geq |P(c)|$ on $(c, a + \delta)$. Since the inequality must hold even for $t = a + \delta$, then $P(a + \delta) \neq 0$.

Now let $\Delta$ be the lub of the set of $\delta$'s such that $0 < \delta < b - a$ and for which (5) holds on $(a, a + \delta)$. Clearly (5) holds on $(a, a + \Delta)$, and so (by the above argument) (5) holds at $t = a + \Delta$. Unless $\Delta = b - a$, the continuity of $P(t)$ gives a contradiction to the lub property of $\Delta$. This completes the proof.

The following theorem, corresponding to Theorem A, is now almost immediate.

**Theorem 1.** If (II) holds for products excluding the $a_{ii}(t)$'s, the problem (3) has a unique solution.

**Proof.** The proof depends only on Lemma 3, and so we must show that Lemma 3 holds even without condition (0). To this end, let $z(t) = G(t)x(t)$, where $x(t)$ satisfies (III) and $G(t)$ is the diagonal matrix for which $g_{ii}(t) = \exp \int_a^t -a_{ii}(s) ds$. Then $z' = B(t)z$, where $b_{ii}(t) = 0$ and, for $i \neq j$, $b_{ij}(t) = a_{ij}(t) \exp \int_a^t [a_{jj}(s) - a_{ii}(s)] ds$. Thus if (II) holds for
A(t) then (II) holds for B(t), with the same $s_{KN}$'s; if (III) holds for $x(t)$ then (III) holds for $z(t)$; and (0) holds for $B(t)$. Thus Lemma 3 as it stands applies to $z(t)$, and also, since $x_i(t) = z_i(t) \exp \int_a^t a_{ii}(s) \, ds$, to $x(t)$. Thus condition (0) can be eliminated from Lemma 3, and the proof is complete.

The theorem corresponding to the dual of Theorem A is contained in the following statements. Corresponding to (I), (II), and (III) we have:

(I') As (I), except that the products with an odd number of factors and those with an even number of factors differ in sign, and there is at least one product, say with $r$ factors ($r \equiv m_K$), which is nonzero at $b$.

Let $(-1)^r s_{KN}$ be 1 or $-1$ according as this last product is positive or negative at $b$. Let $s_{KN} = 1$.

(II') For some fixed $N$, (I) holds for each $K \neq N$ $(1 \leq K \leq n)$; and $x_i(t) = \delta_{iN} (1 \leq i \leq n)$.

(III') $x' = Ax$, and $x_i(b) = \delta_{iN} (1 \leq i \leq n)$.

Lemma 2'. If (0), (I') and (II') hold, there is a $\delta > 0$ such that $s_{KN} x_K(t) > 0$ on $(b - \delta, b)$.

Proof. The proof is like that of Lemma 2, using (4). Alternatively, let $A(t) = A(b)$ for $t > b$; let $s = 2b - t$, $B(s) = -A(t)$, and $w(s) = x(t)$; and apply Lemma 2 directly to the system $w'(s) = B(s)w(s)$ on the interval $[b, 2b - a]$.

Lemma 3'. If (0), (I') and (II') hold, then

$$s_{iN} x_i(t) > 0 \quad (1 \leq i \leq n)$$

holds on $[a, b]$.

Theorem 1'. If (II') holds for products excluding the $a_{ii}(t)$'s, the problem

$$y' = Ay + f, \quad Qy(a) + Py(b) = \beta$$

has a unique solution.

Theorem 1 implies a stronger version of Theorem A, since the hypotheses of Theorem A imply (II) for each $N$. Further, coefficient matrices with vanishing entries can be treated; in particular, known theorems (e.g., see [2] and [3]) for the $n$th order scalar case are implied.

3. The three-point problem. Let $a < c < b$.

Theorem 2. Let $M$ and $N$ be fixed $(1 \leq M \leq n, 1 \leq N \leq n, M \neq N)$.
Let (II) hold on \([c, b]\) and (II') hold on \([a, c]\) for products excluding the \(a_i(t)\)'s, for \(N\) and also for \(M\), with \(s_{MN} = -1\). Let \(R = (\delta_{iM} \delta_{jM})\), \(Q = (\delta_{iN} \delta_{jN})\), and \(S = E - R - Q\). Then the problem

\[
y' = Ay + f, \quad Qy(a) + Sy(c) + Ry(b) = \beta
\]

has a unique solution.

**Proof.** Let \(X' = AX, X(c) = E\); it suffices to show that \(QX(a) + SX(c) + RX(b)\) is nonsingular, the determinant in question being \(x_{NN}(a)x_{MM}(b) - x_{NM}(a)x_{MN}(b)\). By Lemma 3, \(x_{MM}(b)\) and \(s_{MN}x_{MN}(b)\) are positive; by Lemma 3', \(x_{NN}(a)\) and \(s'_{NM}x_{NM}(a)\) are positive; hence the determinant is positive.

It is a matter of detail to verify that the hypotheses of Theorem B imply those of Theorem 2. As in the two-point case, coefficient matrices with vanishing entries, and in particular the scalar case (e.g., Theorem 2 in [2]), are allowed.

**References**


**University of Utah**