A SIMPLE SET WHICH IS NOT EFFECTIVELY SIMPLE

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For each e, let f_e be the partial recursive function

\[ U(\mu y T_1(e, n, y)) \]

and let W_e be the range of f_e. Then W_0, W_1, W_2, \ldots is the Kleene enumeration of the recursively enumerable sets. Post [5] calls a recursively enumerable set simple if its complement is infinite but does not contain any infinite, recursively enumerable set. Raymond Smullyan calls a recursively enumerable set W effectively simple if its complement is infinite, and if there is a partial recursive function f such that for each e, if W_e is contained in the complement of W, then f(e) is defined and is greater than the cardinality of W_e. Clearly, an effectively simple set is simple. The simple set S constructed by Post in [5] is effectively simple. This latter is no accident. In fact it is not unreasonable to claim that any direct attack on the problem of constructing a simple set must result in an effectively simple set. Our purpose here is to obtain a simple set which is not effectively simple. We will make strong use of the recursion theorem of Kleene [2]; however, we will use it in the informal manner of Myhill [4]. Our notation is that of [2].

We introduce a recursive function E:

\[ E(0) = \mu x T_1((x)_0, (x)_1, (x)_2); \]
\[ E(s + 1) = \mu x [x > E(s) & T_1((x)_0, (x)_1, (x)_2)]. \]

We will need E to simultaneously enumerate all the recursively enumerable sets in a fashion suitable for the proving of our theorem. It is a peculiarity of our proof that we cannot rely merely on the usual properties associated with any standard enumeration of the recursively enumerable sets; instead, we are forced to specify a particular enumeration. For each e and s we define a finite set W^s_e: for each m, m \in W^s_e if and only if for some i \leq s,

\[ m = U((E(i))_2) & e = (E(i))_0. \]

Then for each e, \( W^0_e \subseteq W^1_e \subseteq W^2_e \subseteq \cdots \), and \( W_e = \bigcup \{ W^s_e | s \geq 0 \} \). We
say $s$ defines $f_e(n)$ if $e = (E(s))_s$ and $n = (E(s))_t$. If $s$ defines $f_e(n)$, then $f_e(n) = U((E(s))_s)$. For each $e$ and $n$, let

$$S(e, n) \simeq \mu s \quad (s \text{ defines } f_e(n)).$$

$S$ is partial recursive, and $S(e, n)$ is defined if and only if $f_e(n)$ is defined.

Let $0$ denote the empty set. It is clear there exists a recursive function $g$ such that for each $e$, $i$ and $z$, we have

$$W_{g(e, i, z)} = \left\{ \begin{array}{ll} \{ 2^i \cdot 3^i \mid E(S(e, z)) < t \leq E(S(e, z)) + f_e(z) \} & \text{if } f_e(z) \text{ is defined}, \\
0 & \text{otherwise.} \end{array} \right.$$ 

The recursion theorem tells us that there exists a recursive function $z$ such that for each $e$ and $i$, we have

$$W_{z(e, i, z)} = W_{g(e, i, z(e, i))} = \left\{ \begin{array}{ll} \{ 2^i \cdot 3^i \mid E(S(e, z(e, i))) < t \leq E(S(e, z(e, i))) + f_e(z(e, i)) \} & \text{if } f_e(z(e, i)) \text{ is defined}, \\
0 & \text{otherwise.} \end{array} \right.$$ 

We note some properties of $z$:

1. If $f_e(z(e, i))$ is defined, then $f_e(z(e, i))$ is equal to the cardinality of $W_{z(e, i)}$;
2. If $i \neq j$, then $W_{z(e, i)} \cap W_{z(e, j)} = 0$;
3. If $f_e(z(e, i))$ is defined, then for all $n$, $W_{z(e, z)} \cap W_{z(e, i)} = 0$;
4. If $i \neq j$ and both $f_e(z(e, i))$ and $f_e(z(e, j))$ are defined, then $z(e, i) \neq z(e, j)$.

To prove (3), let $s = S(e, z(e, i))$ and let $m \in W_{z} \cap W_{z(e, i)}$. Then

$$m \leq E(s),$$

since $m = U((E(t))_s)$ for some $i \leq s$, and since $E$ is an increasing function. (Recall that $U(x) \leq x$ for all $x$.) But $m > E(s)$, since $m = 2^i \cdot 3^t$ for some $t > E(s)$.

**Theorem 1.** There exists a simple set which is not effectively simple.

**Proof.** We will define a sequence $A$, $B$, $Q_0$, $Q_1$, $Q_2$, \ldots of simultaneously recursively enumerable sets. $A$ will be simple, but not effectively simple. $B$ will be such that if $e \in B$, then $W_e \cap A \neq 0$. We will see to it that if $W_e$ is finite, then $e \in B$. Each $Q_t$ will be finite and will contain a set that will serve as a witness to the fact that $f_e$ does not effectively bound the cardinalities of the finite subsets of the complement of $A$.

Stage $s = 0$. We set $A^0 = B^0 = Q_t^i = 0$ for all $i$. 

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Stage $s > 0$. Let $e = (E(s))_0$ and $n = (E(s))_1$. Thus $s$ defines $f_s(n)$. We perform the following two operations in the indicated order:

(a) We set $Q'_j = Q^{-1}_j$ for all $j \neq e$. If there is no $i$ such that $i \leq e$ and $n = z(e, i)$, we set $Q'_j = Q^{-1}_j$. If there is such an $i$, then by (4) it is unique. In addition, $S(e, z(e, i))$ is defined and

$$W_{s(e, i)} = \{2^i \cdot 3^t \mid E(S(e, z(e, i))) < t \leq E(S(e, z(e, i))) + f_s(z(e, i))\}.$$

We set $Q'_j = Q^{-1}_j \cup W_{s(e, i)}$.

(b) If $e \in B^{\ast - 1}$ or if there is no $m$ such that

$$m \in W^*_s \& (j)_{j \leq e} (m \in Q'_j),$$

then we set $B^* = B^{\ast - 1}$ and $A^* = A^{\ast - 1}$. If $e \in B^{\ast - 1}$ and there is an $m$ with the above property, let $i$ be the least one. We set $B^* = B^{\ast - 1} \cup \{e\}$ and $A^* = A^{\ast - 1} \cup \{i\}$.

Let $A = U \{A^* \mid s \geq 0\}$ and $B = U \{B^* \mid s \geq 0\}$. Since $E$ and $z$ are recursive, it follows $A$ is recursively enumerable. For each $e$, let

$$Q_e = U \{Q^*_i \mid s \geq 0\}.$$

$L_e$ is finite; in fact,

$$L_e = U \{W_{s(e, i)} \mid i \leq e\},$$

since $Q^{-1}_e \neq Q^*_e \cup W_{s(e, i)}$ if and only if $i \leq e$ and $s = S(e, z(e, i))$.

Lemma 1. If $W_e$ is infinite, then $A \cap W_e \neq 0$.

Proof. We know $Q_j$ is finite for every $j$. Let $m$ be a member of $W^*_e$ which is greater than every member of $Q_j$ for all $j \leq e$. Let $s$ be such that $m \in W^*_s$. First we suppose $e \in B^{\ast - 1}$. Then there must be a $t < s$ such that $e \in B^{t - 1}$ and $e \in B^t$. At stage $t$ we must have performed operation (b) in such a manner that $B^t = B^{t - 1} \cup \{e\}$ and $A^t = A^{t - 1} \cup \{i\}$, where $i \in W^*_e$. Now we suppose $e \in B^{\ast - 1}$. We have

$$m \in W^*_s \& (j)_{j \leq e} (m \in Q'_j).$$

But then operation (b) at stage $s$ forces us to put a member of $W^*_e$ in $A^*$. 

Lemma 2. If $m \in W^*_e - Q'_j$, then $m \in Q_j$.

Proof. Suppose for the sake of a reductio ad absurdum that $m \in W^*_e - Q'_j$, and $m \in Q_j$. Since $Q_i = U \{W_{s(j, i)} \mid i \leq j\}$, there must be an $i \leq j$ such that $m \in W_{s(j, i)}$. Since $W_{s(j, i)}$ is nonempty, $f_j(z(j, i))$ is defined. Let $t = S(j, z(j, i))$. Then $t$ defines $f_j(z(j, i))$, and consequently,

$$Q^t_j = Q^{t - 1}_j \cup W_{s(j, i)}.$$
since $i \leq j$. Since $m \in Q_i$, we must have $s < t$. Since $m \in W_x$, we have

$$m \in W_x^{(i)} \cap W_s^{(j,i)} \neq 0.$$  

But this last contradicts (3).

**Lemma 3.** If $m \in Q_i \cap A$, then there exists an $s$ and an $e$ such that $(E(s))_0 = e < i$ and $\{e\} = B^s - B^{e-1}$ and $\{m\} = A^s - A^{e-1}$.

**Proof.** Since $m \in A$, there is an $s$ such that $\{m\} = A^s - A^{s-1}$. Let $e = (E(s))_0$. Since $A^s \neq A^{e-1}$, we must have $\{e\} = B^s - B^{e-1}$. In addition,

$$m \in W_x^s \cap (j)_{s \leq e} (m \in Q_i).$$

It follows from Lemma 2 that $(j)_{s \leq e} (m \in Q_i)$. But then $e < i$, since $m \in Q_i$.

**Lemma 4.** The set $Q_i \cap A$ has at most $i$ members.

**Proof.** Suppose $m$ and $n$ are distinct members of $Q_i \cap A$. Lemma 3 guarantees the existence of $s(m)$, $e(m)$, $s(n)$ and $e(n)$ with properties as stated in the conclusion of Lemma 3. Thus

$$\{m\} = A^{s(m)} - A^{s(m)-1} \cup \{n\} = A^{e(n)} - A^{e(n)-1}.$$  

Since $m \neq n$, it follows $s(m) \neq s(n)$. But then $e(m) \neq e(n)$, since

$$\{e(m)\} = B^{s(m)} - B^{s(m)-1} \cup \{e(n)\} = B^{e(n)} - B^{e(n)-1}.$$  

We also know from Lemma 3 that $e(m) < i$ and $e(n) < i$. Thus we can map the set $Q_i \cap A$ in a one-to-one fashion into the set $\{e \mid e < i\}$.

**Lemma 5.** For each $e$, there is a $z$ such that $W_x$ is contained in the complement of $A$ and such that either $f_x(z)$ is undefined or $f_x(z)$ is not greater than the cardinality of $W_x$.

**Proof.** Fix $e$. We show that some member of the sequence, $z(e, 0)$, $z(e, 1)$, $\cdots$, $z(e, e)$ serves as the desired $z$. Suppose there is an $i \leq e$ such that $f_x(z(e, i))$ is undefined. Then $W_x^{(e, i)} = 0$, and the lemma is proved. Suppose then that $f_x(z(e, i))$ is defined for all $i \leq e$. By (1), $f_x(z(e, i))$ is not greater than the cardinality of $W_x^{(e, i)}$ for any $i \leq e$. Thus it suffices to find an $i \leq e$ such that $W_x^{(e, i)} \cap A = 0$. The sets,

$$W_x^{(e, 0)}, W_x^{(e, 1)}, \cdots, W_x^{(e, e)}$$

are nonempty and disjoint. If each of them has a member in $A$, then their union has at least $e + 1$ members in $A$. But their union is $Q_e$, and Lemma 4 tells us that $Q_e$ has at most $e$ members in $A$.

It follows from Lemma 5 that $A$ is not effectively simple. It also
follows from Lemma 5 that the complement of $A$ is infinite, since otherwise, the constant function

$$f(n) = 1 + \text{cardinality of the complement of } A$$

would constitute a counterexample to Lemma 5. Finally, by Lemma 1, $A$ is simple.

Post [5] calls a recursively enumerable set $W$ hyper-simple if its complement is infinite, and if there does not exist a recursively enumerable sequence of disjoint, finite sets, each one of which contains a member of the complement of $W$. It can be shown with the help of Lemma 4 that $A$ is not hyper-simple.

The proof of Theorem 1 above is, as far as we know, the first proof in recursion theory to make simultaneous use of the recursion theorem and the priority method of Friedberg [1] and Muchnik [3]. The priority method was needed to resolve the inevitable conflict between putting elements in $A$ as required by Lemma 1 and keeping them out of $A$ as required by Lemma 4. Thus in operation (b), we are not allowed to take $m$ from $W_{e}$ and add it to $A^{*}$ if for some $j \leq e$, $m \in Q_{j}$. The recursion theorem was needed to prove that our system of priorities does eventually resolve all conflicts happily; in particular, the recursion theorem made possible the proof of Lemma 2.

References


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