A SIMPLY CONNECTED 3-MANIFOLD IS \(S^3\) IF IT IS THE SUM OF A SOLID TORUS AND THE COMPLEMENT OF A TORUS KNOT

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It has been shown [6] that any closed, connected, orientable 3-manifold can be constructed by removing a finite number, \(k\), of mutually exclusive tame solid tori\(^1\) from the 3-sphere, \(S^3\), and then sewing them back in some possibly different manner. In particular, any homotopy 3-sphere can be obtained in this manner; thus it gives an approach to the Poincaré Conjecture.

We wish to consider the case \(k=1\). To this end let \(T\) be a tame solid torus in \(S^3\) and let \(M\) be a simply connected 3-manifold containing a solid torus \(S\) such that there is a homeomorphism \(h\) of \(S^3-\text{Int}(T)\) onto \(M-\text{Int}(S)\). It is known [1] that \(M\) is \(S^3\) if \(T\) is unknotted or if \(T\) has the type of a trefoil knot. In [5] it is asserted that \(M\) is \(S^3\) in every case. The proof given in [5] shows only that \(h\) must take some longitude of \(T\) onto a longitude of \(S\)—a fact that is not sufficient to prove this assertion. The purpose of this paper is to provide a proof that \(M\) is \(S^3\) in the case that \(T\) has the type of a torus knot.

We first give a canonical description of the torus knot of type \((p, q)\). We consider \(S^3\) as the one point compactification of \(E^3\) which we represent in cylindrical coordinates \((r, \theta, z)\). Let \(R\) be the torus \(R-\{(r, \theta, z): (r-2)^2+z^2=1\}\). Let \(p\) and \(q\) be relatively prime positive integers. We assume, for convenience, that \(p>q\); since it is known that the torus knots of types \((p, q)\) and \((q, p)\) are equivalent. For \(1 \leq k \leq p\) let \(x_k = (3, 2k\pi/p, 0)\) and \(y_k = (1, 2k\pi/p, 0)\). Let \(a_k\) be an arc on \(R\) from \(x_k\) to \(y_{k+q}\) that lies in the positive \(z\) half space and which increases monotonically with \(\theta\). (It is to be understood that the subscripts are to be reduced modulo \(p\).) Let \(b_k\) be the arc on \(R\) from \(x_{k+q}\) to \(y_{k+q}\) that lies in the negative \(z\) half space and the plane \(\theta=2(k+q)\pi/p\). We denote the simple closed curve \(U_{k,q}^p (a_k \cup b_k)\) by \(K_{p,q}\). Figure 1 shows \(K_{6,3}\).

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\(^1\) A solid torus \(T\) is a set homeomorphic to the Cartesian product of a disk and a simple closed curve. Any simple closed curve on \(\text{Bd} \ T\) which bounds a disk in \(T\) but which does not separate \(\text{Bd} \ T\) is called a meridian of \(T\). Any simple closed curve on \(\text{Bd} \ T\) which does not separate \(\text{Bd} \ T\), which is not a meridian of \(T\), and which intersects some meridian of \(T\) in exactly one point is called a longitude of \(T\).
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It is well known that any torus knot (i.e., a simple closed curve which lies on a tame, unknotted torus in $S^3$) can be taken onto one of the curves $K_{p,q}$ by a (possibly orientation reversing) homeomorphism of $S^3$ onto itself.

The fundamental group of $S^3 - K_{p,q}$ can be given a presentation in the standard manner (see [4, Chapter III]) with the overcrossings, $a_k$, representing generators and the undercrossings, $b_k$, giving the relations. The convention we adopt is that the generator $a_k$ represents a loop which begins at the base point $x_0$ (which we choose to be obtained by moving $x_1$ slightly in the negative $\theta$ direction as shown in Figure 1), passes under the segment $a_k$ from left to right (in terms of the orientation assigned to $a_k$), and returns to $x_0$. It is understood that this loop passes under no other segment in the projection of the knot.

The relation corresponding to the undercrossing $b_k$ may be read in the standard manner (see Figure 2), and we obtain the following presentation of $\pi_1(S^3 - K_{p,q})$.

\[(1) \quad \{a_1, a_2, \ldots, a_p; a_k a_{k+1} \ldots a_{k+q-1} = a_{k+1} a_{k+2} \ldots a_{k+q}\}, \]

\[k = 1, 2, \ldots, p.\]
If we let $X = a_1a_2 \cdots a_q$, $Y = a_1a_2 \cdots a_p$, we show in the following manner that $X$ and $Y$ generate $\pi_1(S^3-K_{p,q})$. Since $p$ and $q$ are relatively prime, there are integers $r$ and $s$ such that $rp + sq = 1$. If $r > 0$ (hence $s < 0$), then

$$Y^r = (a_1a_2 \cdots a_p)(a_1a_2 \cdots a_p) \cdots (a_1a_2 \cdots a_p)$$

$$= a_1(a_2a_3 \cdots a_{q+1})(a_{q+2}a_{q+3} \cdots a_{q+1}) \cdots (a_{p-q+1}a_{p-q+3} \cdots a_p)$$

$$= a_1X^{-1}.$$

While if $r < 0$ (hence $s > 0$), then

$$X^s = (a_1a_2 \cdots a_q)(a_{q+1}a_{q+2} \cdots a_{2q}) \cdots (a_{p-q+1}a_{p-q+3} \cdots a_p)$$

$$= (a_1a_2 \cdots a_p)(a_1a_2 \cdots a_p) \cdots (a_1a_2 \cdots a_p)a_1$$

$$= Y^{-r}a_1.$$

In either case we see that $a_1 = Y^rX^s$. By the same technique we can show that $a_2 = X^{-r}Y^sX^s_1 \cdots a_k = X^{(1-k)r}Y^sX^{ks}$. Using these relationships it can be shown that $\pi_1(S^3-K_{p,q})$ has the following presentation in terms of the elements $X$ and $Y$:

\[
\{ X, Y: X^p = Y^q \}.
\]

To verify this one need only show that the maps $\phi: (1) \mapsto (2); \psi$ generated by $\phi(a_k) = X^{(1-k)r}Y^sX^{ks}$, $\psi(X) = (a_1a_2 \cdots a_q)$, and $\psi(Y) = (a_1a_2 \cdots a_p)$ preserve relations and that $\phi\psi = 1$ and $\psi\phi = 1$.

We are now prepared to prove our result. First we state the following well-known

**Lemma.** Suppose $M$ and $N$ are 3-manifolds containing solid tori $T$ and $S$, respectively, and $h$ is a homeomorphism of $M-\text{Int}(T)$ onto $N-\text{Int}(S)$ which takes $\text{Bd} \ T$ onto $\text{Bd} \ S$ and which takes some meridian of $T$ onto a meridian of $S$. Then $h$ can be extended to a homeomorphism of $M$ onto $N$. 


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THEOREM. Suppose $T$ is a tubular neighborhood of the torus knot $K_{p,q}$, $M$ is a 3-manifold, and $S$ is a solid torus in $M$ such that $S^3 - \text{Int}(T)$ is homeomorphic to $M - \text{Int}(S)$. Then if $M$ is simply connected, $M$ is homeomorphic to $S^3$.

PROOF. We assume without loss of generality that $T \cap R$ is an annular neighborhood of $K_{p,q}$ on $R$ one of whose boundary components, which we denote by $\alpha$, contains the point $x_0$. By choosing the proper orientation for $\alpha$ we see that $\alpha$ represents the element $(a_1a_2 \cdots a_p)^a = Y^a$ of $\pi_1(S^3 - \text{Int}(T))$. We let $\beta$ be the component of the intersection of the plane $z = 0$ with $\partial T$ which contains $x_0$. Properly oriented, $\beta$ represents the element $a_1$ of $\pi_1(S^3 - \text{Int}(T))$; furthermore $\alpha$ and $\beta$ serve as generators for the abelian group $\pi_1(\partial T)$.

Now let $h: S^3 - \text{Int}(T) \to M - \text{Int}(S)$ be the homeomorphism given by the hypothesis. Let $\gamma$ be a meridian of $S$ containing the point $h(x_0)$. Now $h^{-1}(\gamma)$ represents an element of the form $\alpha^m\beta^n = (a_1a_2 \cdots a_p)^{m+n} = Y^m(Y^*X^*)^n$. We obtain $\pi_1(M)$ from $\pi_1(S^3 - \text{Int}(T))$ by adjoining the additional relation $h^{-1}(\gamma) = 1$. This yields the following:

\[(3) \quad \pi_1(M) = \{X, Y : X^p = Y^q, Y^m(Y^*X^*)^n = 1\}.
\]

We will show that $m = 0$, $n = \pm 1$; hence that $h^{-1}(\gamma)$ is a meridian of $T$. In light of the previous lemma, this will complete the proof.

The commutator quotient group of $\pi_1(M)$ has the presentation $\{a_1 : a_1^{2m+n} = 1\}$. Since this group is trivial, we must have $n = \pm 1 - pqm$. Now neither $p$ nor $q$ is 1; for otherwise $T$ would be unknotted. Suppose that $m \neq 0$. Then $n \neq 0, \pm 1$. We will show that this assumption leads to a contradiction as follows.

Consider the group:

\[G_{p,q,n} = \{R_1, R_2, R_3 : R_1^2 = R_2^2 = R_3^2 = (R_1R_2)^p = (R_2R_3)^p = (R_1R_3)^p = (R_1R_4)^n = 1\}.
\]

If $p, q, n \neq 0, \pm 1$, this group is nontrivial [2, p. 55]. $G_{p,q,n}$ may be represented as follows. Let $PQN$ be a triangle with angles $\pi/|p|, \pi/|q|, \pi/|n|$ in the hyperbolic plane, the plane, or the 2-sphere according as $\pi/|p| + \pi/|q| + \pi/|n|$ is less than, equal to, or greater than $\pi$. We let $R_1, R_2,$ and $R_3$ be reflections through the "lines" on $QN, PQ,$ and $PN,$ respectively. Then $R_1R_2, R_2R_3,$ and $R_3R_2,$ respectively, are rotations of angles $2\pi/|q|, 2\pi/|p|,$ and $2\pi/|n|$ about the vertices $Q, P,$ and $N$. We can obtain a nontrivial representation of $\pi_1(M)$ into $G_{p,q,n}$ given by $\eta(X) = (R_2R_3)^p, \eta(Y) = (R_2R_3)^p$. Note that $\eta(X^p) = (R_2R_3)^{p^2} = 1 = (R_1R_2)^{p^2} = \eta(Y^p)$, and that $\eta(Y^m(Y^*X^*)^n)$
= (R_1R_3)^n(\eta(R_1R_3)(R_2R_3)^n) = ((R_1R_3)^{1-n}(R_2R_3)^{1-n})^n = (R_1R_2R_3)^n
= (R_1R_3)^n = 1. Hence \eta is a homomorphism. This gives the contradiction that \tau_1(M) is nontrivial. Thus we have m = 0. Since \gamma is a simple closed curve, we must have n = \pm 1, and the proof is complete.

Note that the homeomorphism between $S^3$ and $M$ given by the conclusion of the above theorem was an extension of the homeomorphism between $S^3 - \text{Int}(T)$ and $M - \text{Int}(S)$ given by the hypothesis. From this observation we can state the following as a direct consequence of the above theorem. (It was pointed out by the referee that this corollary is a known result. See [3, p. 183].)

**Corollary.** Suppose $T$ is a tubular neighborhood of the torus knot $K_{p,q}$ and $h$ is a homeomorphism of $S^3 - \text{Int}(T)$ onto a tame subset of $S^3$. Then $h$ can be extended to a homeomorphism of $S^3$ onto itself.

**References**

1. R. H. Bing, Necessary and sufficient conditions that a 3-manifold be $S^3$, Ann. of Math. (2) 68 (1958), 17–37.


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