MULTIPLICATIVE FUNCTIONALS OF A MARKOV PROCESS

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The present note may best be viewed as an addendum to Meyer's important paper [2]. As such we refer the reader to [2] for all notations, definitions, etc. In particular \( \{X_t\} \) will always denote a temporally homogeneous Markov process with state space \( X \) (locally compact, separable) satisfying Hunt's hypothesis (A). See [1] or [2].

Let \( \{M_t\} \) be a normalized multiplicative functional of \( \{X_t\} \) [2, p. 136 and p. 141], then Meyer [2, p. 153] has obtained the following very important result: \( \{M_t\} \) has the strong Markov property, that is, for each stopping time \( T \) and random variable \( R \geq 0 \) one has

\[
M_{R+T}(\omega) = M_T(\omega)M_R(\theta_T\omega)
\]
a.s., \( P^\omega \) for each \( \omega \) in \( X \). Unfortunately Meyer's proof contains a slight gap (in the proof of Theorem 4.2 on p. 152 of [2]) and the result is not valid without additional assumptions on \( \{M_t\} \) as the following example shows. Let \( \{X_t\} \) be one-dimensional Brownian motion (so that \( X \) is the real line) and define \( M_t(\omega) = 1 \) for all \( t \geq 0 \) if \( X_0(\omega) \neq 0 \) and \( M_t(\omega) = 0 \) for all \( t \geq 0 \) if \( X_0(\omega) = 0 \). It is easy to see that this defines a multiplicative functional of \( \{X_t\} \) since \( P^\omega(X_t = 0) = 0 \) for all \( t > 0 \) and \( x \) in \( X \). On the other hand if \( T \) is the first passage time to 0 it is immediate that (1.1) is not valid.

We state the following criterion for the strong Markov property.

**Theorem.** The normalized multiplicative functional \( \{M_t\} \) has the strong Markov property if and only if for every stopping time \( T \) and \( x \) in \( X \),

\[
P^\omega[X_T \in N, M_T > 0] = 0,
\]

where \( N \) is the (universally measurable) set of nonpermanent points of \( \{M_t\} \), i.e., \( N = \{x : P^\omega(M_0 = 0) = 1\} \).

**Proof.** The sufficiency may be established exactly as in [2], the condition (1.2) being just the condition necessary to make the proof of Theorem 4.2 [2, p. 152] valid. (One can give a much simpler proof of the sufficiency using resolvents instead of semi-groups.) To prove the necessity let \( R = \inf\{t : M_t = 0\} \), then \( R \) is a stopping time and the right continuity of \( \{M_t\} \) implies that \( P^\omega(M_R > 0, R < \infty) = 0 \) for all \( x \). Let \( T \) be any stopping time and consider the random variable

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If \( x \) is fixed the strong Markov property yields

\[
E^*[M_H, H < \infty] \leq E^*[M_T X^r(M_T, R < \infty)] = 0.
\]

Thus \( H = T + R(\theta_T) \geq R \) on \( \{ H < \infty \} \) and so \( T + R(\theta_T) \geq R \), both statements holding a.s., \( P^x \). Therefore

\[
P^x[X_T \in N, M_T > 0] = P^x[X_T \in N, T < R] \leq E^*[P^x(T)(R > 0), X_T \in N] = 0,
\]

since \( P^y (R>0) = 0 \) for all \( y \) in \( N \).

If \( T = \inf \{ t > 0, X_t \in N \} \) is a stopping time, then (1.2) can be replaced by \( P^x(M_T > 0, T < \infty) = 0 \) for all \( x \). This is the case if \( N \) is nearly analytic. The multiplicative functional defined in the second paragraph gives an example of a nonperfect multiplicative functional since any perfect multiplicative functional has the strong Markov property. (See [2, p. 136] for the definition of a perfect multiplicative functional.) As an example of a strongly Markov multiplicative functional for which the corresponding semi-group \( \{ Q_t \} \) is not exactly subordinate to \( \{ P_t \} \) (the semi-group of \( \{ X_t \} \)), let \( \{ X_t \} \) be translation to right along the real line at unit speed. Put \( M_t(\omega) = 0 \) if \( t + X_0(\omega) \geq 0 \) and \( X_0(\omega) \leq 0 \), and \( M_t(\omega) = 1 \) otherwise. It is easily verified that this example has the desired properties.

REFERENCES


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