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A NOTE ON THEOREMS OF BURNSIDE AND BLICHFELDT

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1. The irreducible constituents of the tensor powers of a representation of a group. In his book [3], W. Burnside proved the following theorem (Theorem IV of Chapter XV):

THEOREM 1. *Let G be a finite group and let X be a faithful representation of G over the field $\mathbf{C}$ of complex numbers. Each irreducible representation X_λ of G appears as a constituent in some tensor power of X.*

Recently, R. Steinberg [5] has given a very simple proof of this theorem generalizing it at the same time. I shall give still another proof of Theorem 1. While this proof is less conceptual than Steinberg's proof, it is very short, and it refines the theorem in another direction.

THEOREM 1*. *Assume that the character χ of the representation X in Theorem 1 takes on a total of r distinct values $a_1, a_2, \cdots, a_r$ on G. Each irreducible character χ_λ of G appears as a constituent of one of the characters $\chi^0 = 1, \chi, \chi^2, \cdots, \chi^{r-1}$.*

PROOF. Let A_j be the set of elements $g \in G$ for which $\chi(g) = a_j$. Choose $g_j \in A_j$. If χ_λ is not contained in χ^i, then

$$|G| (\chi^i, \chi_\lambda) = \sum_j \chi^i(g_j) \sum_{g \in A_j} \bar{\chi}_\lambda(g) = 0.$$

If this holds for $i = 0, 1, \cdots, r-1$, it follows from the nonvanishing

of the Vandermonde determinant that the inner sum vanishes for all j. Since X is faithful, one of the A_j consists only of the unit element, and we would have $D_\theta \chi_\lambda = \chi_\lambda(1) = 0$, which is absurd.

2. **Remarks.** 1. If $n = D_\theta \chi$, it follows from Theorem 1* that the sum of the degrees of the irreducible representations of G is at most equal to

$$s = 1 + n + n^2 + \cdots + n^{r-1}.$$

This implies that the order $|G|$ of G is at most equal to s^2.

2. If it is known that all values of χ lie in a fixed algebraic number field K of finite degree over the field Q of rational numbers, then since $|\chi(g)|$ and all its algebraic conjugates are at most equal to n for $g \in G$, it follows that r lies below a bound depending only on K and n. Hence $|G|$ also lies below bounds depending only on K and n. The existence of such bounds has first been observed by I. Schur [4]. We do not obtain here the sharp values given by Schur.

3. The proof of Theorem 1* shows that instead of $1, \chi, \cdots, \chi^{r-1}$, we may take any r consecutive powers of χ. If χ assumes the value 0, even $r-1$ consecutive powers of χ with positive exponents will do.

4. If C is replaced by an algebraically closed field K of prime characteristic p, the same result will hold for the irreducible representations X_λ of G in K. If we take for χ_λ the character of the indecomposable component of the regular representation, associated with X_λ, the same proof applies. Here, $a_1, a_2, \cdots, a_r$ are to be taken as the values assumed by χ for p-regular elements of G. (See [2] for the basic properties of "modular" characters.)

5. The proof of Theorem 1* remains the same if C is replaced by an algebraically closed field K of characteristic 0. It is immediate that here and in Remark 4 the condition that K be algebraically closed can be dropped.

3. **A result on the conjugates of a character.** We show now

THEOREM 2. *Let χ be an irreducible character of a finite group G. Let Ω be the field obtained by adjoining the values of χ to the field Q of rational numbers and let Γ be the Galois group of Ω over Q. If $\theta_1, \theta_2, \cdots, \theta_n$ is a system of n elements of Γ, we have one of the following two cases.*

Case A. There exists an element $g \in G$ such that $\theta_i \chi(g) \neq \chi(g)$ for $i = 1, 2, \cdots, n$.

Case B. There exists a product $\theta_\alpha \theta_\beta \cdots \theta_\rho$ with $1 \leq \alpha < \beta < \cdots < \rho \leq n$ with an odd number of factors which leaves χ invariant.

PROOF. If ξ is an element of the group ring of Γ over the ring of

rational integers, we define $\xi \cdot \chi$ in the natural manner by linearity. Note that $\theta_i \theta_j = \theta_j \theta_i$. Form

$$P = \prod_i (1 - \theta_i) \cdot \chi.$$

If there exists an element $g \in G$ with $P(g) \neq 0$, we must have $\theta_i \chi(g) \neq \chi(g)$ for all i and this leads to Case A. If P vanishes identically, this yields

$$(1) \qquad \chi + \sum_{i<j} \theta_i \theta_j \chi + \cdots = \sum_i \theta_i \chi + \sum_{i<j<k} \theta_i \theta_j \theta_k \chi + \cdots.$$

It follows that the irreducible character χ appears on the right and this means that we have the result of Case B.

REMARK. It is trivial that Theorem 2 remains valid, if Ω is replaced by a larger field normal over Q.

4. Existence of elements of certain orders in G. It will be convenient to say that a number ω "requires" the mth roots of unity, if ω lies in the field of the mth roots of unity over Q, and if m is the least positive integer for which this is true. If m here is even, it will be divisible by 4. We shall also say that a character χ "*requires*" the mth roots of unity, if all values of χ lie in the field of the mth roots of unity, and if again m cannot be replaced by a smaller integer.

We now give some corollaries of Theorem 2. The first one is due to Blichfeldt [1] and Burnside [3, Theorem X of Chapter XVI].

COROLLARY 1. *Let χ be an irreducible character of a finite group. Let $p_1, p_2, \cdots, p_n$ be distinct primes, and assume that there exist elements $g_1, g_2, \cdots, g_n$ of G such that $\chi(g_i)$ requires the $p_i^{a_i}$th roots of unity for some $a_i > 0$, $i = 1, 2, \cdots, n$. Then G contains elements of order $p_1^{a_1} p_2^{a_2} \cdots p_n^{a_n}$.*

PROOF. Let Ω be the field of the $|G|$th roots of unity over Q (see the Remark in 3). If p_i divides $|G|$ with the exact exponent r_i, $(i = 1, 2, \cdots, n)$, then $a_i \leq r_i$. Let Ω_i be the field of the $(|G|/p_i^{r_i-a_i+1})$st roots of unity over Q. Since $\chi(g_i) \notin \Omega_i$, we can find an element θ_i of the Galois group of Ω over Ω_i such that $\theta_i \chi(g_i) \neq \chi(g_i)$. On the other hand, by construction $\theta_i \chi(g_j) = \chi(g_j)$ for $j \neq i$, $1 \leq j \leq n$. If we now apply Theorem 2, we cannot have Case B since a product $\theta_\alpha \theta_\beta \cdots$ will not leave $\chi(g_\alpha)$ fixed. Hence there exist $g \in G$ such that $\theta_i \chi(g) \neq \chi(g)$ for $i = 1, 2, \cdots, n$. Then $\chi(g) \notin \Omega_i$, and this implies that the order of g is divisible by $p_i^{a_i}$ for $i = 1, 2, \cdots, n$. This proves the corollary.

A similar method yields the slightly more general result.

COROLLARY 2. *Let χ be an irreducible character of the finite group G*

and assume that G contains elements g_1, g_2, $\cdots$, g_n such that $\chi(g_i)$ requires the h_ith roots of unity where the rational integers h_1, h_2, $\cdots$, h_n are pairwise coprime. If $p_i^{a_i}$ is a prime power dividing h_i then G contains elements of order $p_1^{a_1} p_2^{a_2} \cdots p_n^{a_n}$.

Another result of the same type is the following one.

COROLLARY 3. *Let χ be an irreducible character of the finite group G which requires the mth roots of unity. Let m_0 be the product of those prime power factors $p_i^{a_i}$ of m whose exponent a_i is greater than 1. Then G contains elements of order m_0.*

PROOF. Let Ω be the field of the mth roots of unity, and let Ω_i here be the field of the (m/p_i)th roots of unity with p_i ranging over those primes for which $p_i^2 \mid m$. Then Ω is cyclic of degree p_i over Ω_i. Choose θ_i as a generator of the Galois group of Ω over Ω_i. Suppose that a product Π of different θ_i left χ invariant. Since θ_i has order p_i and $(p_i, p_j) = 1$ for $i \neq j$, each factor θ_i appearing in Π leaves χ fixed. Then χ would lie in Ω_i and this is not consistent with the choice of m.

Thus, we have Case A in Theorem 2. If $\theta_i(\chi(g)) \neq \chi(g)$ for all i in question, then the order of g is divisible by $p_i^{a_i}$ for all these i and this proves the corollary.

There are still other results which can be obtained by this type of method, but it does not seem to be worthwhile to go into details. It should be mentioned that we have not used (1) fully in the proof of Theorem 2 and that more precise information is available. For instance, if $n = 3$ in Theorem 2, and if we do not have Case A, then either one of the θ_i leaves χ fixed, or the four elements θ_1^2, θ_2^2, θ_3^2, $\theta_1\theta_2\theta_3$ all leave χ fixed.

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