ON COMPACT COMPLEX COSET SPACES OF REDUCTIVE LIE GROUPS

JUN-ICHI HANO

1. The statement of theorems. Let $G$ be a connected complex Lie group and let $B$ be a closed complex Lie subgroup in $G$. The left coset space $G/B$ is a complex manifold, which will be called a complex coset space. We denote by $B_0$ the identity connected component of $B$ and $U$ the normalizer of $B_0$. The canonical projection $p$ of $G/B$ onto $G/U$ defines a holomorphic fibre bundle, and the complex Lie group $U/B_0$ acts on $G/B$ as the structure group. We denote by $(G/B, p, G/U)$ this holomorphic fibre bundle.

Suppose that the complex coset space $G/B$ is compact. Then, by a recent result of Borel-Remmert [1, Satz 7'], it turns out that the base space $G/U$ is a Kaehler $C$-space, that is, a simply connected compact complex coset space admitting a Kaehler metric, such that the group of isometries is transitive on it. Since $G/U$ is simply connected, $U$ must be connected. The complex coset space of the connected complex Lie group $U/B_0$ by the discrete subgroup $B/B_0$ can be regarded as the standard fibre of $(G/B, p, G/U)$. Making use of this result of Borel-Remmert we derive the following.

Theorem 1. Let $G/B$ be a compact connected complex coset space of a connected complex Lie group $G$ by a closed complex Lie subgroup $B$. Let $U$ be the normalizer of the identity connected component $B_0$ of $B$. If $G$ is a reductive complex Lie group, then the fibre of the holomorphic fibre bundle $(G/B, p, G/U)$ is a compact connected complex coset space of a reductive complex Lie group $U/B_0$ by the discrete subgroup $B/B_0$.

If a compact coset space of a connected reductive real Lie group $G'$ by a closed Lie subgroup $B'$ admits an invariant complex structure, the complex manifold $G'/B'$ can be written as a complex coset space of a connected complex reductive Lie group. Hence, this is a case where we can apply the above theorem. This gives a generalization of a theorem proved by Matsushima [2, Theorem 2].

Let $M$ be a connected compact complex manifold. By a theorem of Bochner-Montgomery, the group of all holomorphic homeomorphisms of $M$ onto itself is a complex Lie group acting on $M$ as a holomorphic transformation group. We denote by $A_0(M)$ the identity

Presented to the Society January 27, 1963; received by the editors November 20, 1962.

1 Partial support by National Science Foundation Grant No. GP-89.
connected component of this complex Lie group. If the group $A_0(M)$ acts on $M$ transitively, $M$ can be expressed as a complex coset space of a connected complex Lie group $A_0(M)$.

**Theorem 2.** Let $M$ be a connected compact coset space, and let $A_0(M)$ be the identity connected component of the complex Lie group of holomorphic homeomorphisms of $M$ onto itself. If a connected reductive (real or complex) Lie subgroup in $A_0(M)$ is transitive on $M$, then $A_0(M)$ is a complex reductive Lie group.

When $M$ is a $C$-space, that is, a simply connected compact complex coset space, the fact that $A_0(M)$ is a reductive complex Lie group, more strongly that, it is locally a direct product of a complex vector group and a connected semi-simple complex Lie group, is proved by Wang [3, Theorem III].

2. The proof of Theorem 1. We shall prove Theorem 1 in a slightly more general form, which is required in the proof of Theorem 2. Using the same notations as in Theorem 1, let $X$ be a closed complex subgroup in $G$ such that $B \subseteq X \subseteq U$. We shall show that if the complex coset space $G/X$ is a Kaehler $C$-space, then the factor group $X/B_0$ is reductive. By the theorem of Borel-Remmert mentioned above, the subgroup $U$ satisfies the hypothesis for the group $X$.

Wang's structure theorem asserts that a complex coset space of a connected complex Lie group $G$ by a closed complex Lie subgroup $X$ is a Kaehler $C$-space if and only if $X$ is connected and contains a maximal connected solvable Lie subgroup in $G$ [3]. Therefore, $X$ is connected and contains the identity connected component $Z$ of the center of $G$. We denote by $g$ the Lie algebra of $G$ and by $u$, $x$, $b$, and $z$ the subalgebras in $g$ corresponding to the complex Lie subgroups $U$, $X$, $B_0$, and $Z$, respectively. We always understand that the base field of those Lie algebras is the field of complex numbers. As $G$ is reductive, $g$ is the direct sum of the center $z$ and the maximal semi-simple ideal $g_1$. We review here a proposition by Wang [3, Proposition 5.2] about the subalgebra $x$ corresponding to the isotropy subgroup $X$ of the Kaehler $C$-space $G/X$. We can choose a Cartan subalgebra $h$ of $g_1$ which is contained in $x$, an ordering of the set of roots with respect to $h$, and a subset $\Delta'$ consisting of some positive roots, so that a root vector $X_\alpha$ belonging to a root $\alpha$ is contained in $x$ if and only if either $\alpha$ is positive or $-\alpha$ is in $\Delta'$. Let $\Delta''$ be the set of the positive roots not contained in $\Delta'$. Then, we have

$$x = z + h + \sum_{\alpha \in \Delta'} \{X_\alpha\} + \sum_{\beta \in \Delta''} \{X_\beta\}.$$
ON COMPACT COMPLEX COSET SPACES

The subspace \( n \) spanned by the \( X_\beta, \beta \in \Delta'' \), is a nilpotent ideal in \( x \), and a factor algebra of \( x \) by an ideal containing \( n \) is always a reductive Lie algebra.

Thus, in order to complete the proof of the statement, it suffices to show that \( b \supset n \). Since \( b \) is an ideal in \( x \), \( x \) is stable by \( \text{ad} \, H, H \in \mathfrak{h} \). This implies that \( b \) is spanned by \( b \cap (x + h) \) and some root vectors. First, we shall show that for a root \( \alpha \) in \( \Delta' \), \( X_\alpha \) belongs to \( b \) if and only if \( X_{-\alpha} \) belongs to \( b \). Take a root vector \( X_\alpha \) such that \( X_\alpha \in b \) and that either \( \alpha \) or \(-\alpha \) belongs to \( \Delta' \). Then, \([X_{-\alpha}, X_\alpha] \in b \cap h \) and \([X_{-\alpha}, X_\alpha], X_{-\alpha}\] = \( \alpha ([X_{-\alpha}, X_\alpha]) \cdot X_{-\alpha} \in b \). As is known in the theory of a semi-simple complex Lie algebra, the complex number \( \alpha ([X_{-\alpha}, X_\alpha]) \) never vanishes, which implies that \( X_{-\alpha} \in b \). Let us denote by \( \text{ad}_b \, H \) and \( \text{ad}_x \, H \) the restrictions of \( \text{ad} \, H \) to the subspaces \( b \) and \( x \), respectively, and by \( \Delta \) the set of the roots \( \alpha \) such that \( \alpha \in \Delta'' \) and that \( X_\alpha \in b \). Then, from the fact shown above it follows that for any \( H \in \mathfrak{h} \),

\[
\text{trace} (\text{ad}_x \, H) - \text{trace} (\text{ad}_b \, H) = \sum_{\alpha \in \Delta} \alpha(H).
\]

Since the coset space \( X/B \) of \( X/B_0 \) by the discrete subgroup \( B/B_0 \) is compact, the factor group \( X/B_0 \) is a unimodular Lie group. Therefore, the trace of the linear transformation \( \text{ad} \, H, H \in \mathfrak{h} \), must vanish, and accordingly the sum of all roots in \( \Delta \) is equal to zero. On the other hand, as every root in \( \Delta \) is positive, we see that the set \( \Delta \) is empty. This implies that \( b \) contains \( n \), completing the proof.

3. The proof of Theorem 2. We denote by \( \overline{G}, \overline{B} \) the group \( A_0(M) \), its isotropy subgroup at a point in \( M \), respectively. Let \( \overline{U} \) be the normalizer of the identity component \( B_0 \) of \( \overline{B} \). In virtue of the theorem of Borel-Remmert mentioned above, \( \overline{G}/\overline{U} \) is a Kaehler C-space and \( \overline{U} \) is connected. We denote by \( \overline{\varphi} \) the canonical projection of \( \overline{G}/\overline{B} \) onto \( \overline{G}/\overline{U} \).

Let \( G \) be the least connected complex Lie subgroup in \( \overline{G} \) containing the given connected reductive Lie subgroup which is transitive on \( \overline{G}/\overline{B} \). Obviously, \( G \) is also reductive and is transitive on \( \overline{G}/\overline{B} \). The isotropy subgroup \( B \) in \( G \) is \( G \cap \overline{B} \). The group \( G \) acts on \( \overline{G}/\overline{U} \) transversely and its isotropy subgroup \( X \) is equal to \( G \cap \overline{U} \). Since \( \overline{U} \) is the normalizer of \( B_0, X \) is contained in the normalizer \( U \) in \( G \) of the identity connected component \( B_0 \) of \( B \). The complex coset space \( G/X \), which coincides with \( \overline{G}/\overline{U} \), is a Kaehler C-space. By Theorem 1, the factor group \( X/B_0 \) is a connected reductive complex Lie group. The fibre \( \overline{U}/\overline{B} \) of the holomorphic fibre bundle \( (\overline{G}/\overline{B}, \overline{\varphi}, \overline{G}/\overline{U}) \) is equal to \( X/B \); indeed, we have a holomorphic homeomorphism of \( X/B \) onto
$\mathcal{U}/\mathcal{B}$, which is induced from the injection of $X$ into $\mathcal{U}$. Moreover, we see that the homomorphism of $X/B_0$ into $\mathcal{U}/\mathcal{B}_0$ induced from the injection of $X$ into $\mathcal{U}$ is locally isomorphic and onto. This follows from the fact that the dimensions of $X/B_0$ and $\mathcal{U}/\mathcal{B}_0$ are equal to those of $X/B$ and $\mathcal{U}/\mathcal{B}$, respectively. Thus, $\mathcal{U}/\mathcal{B}_0$ is a reductive complex Lie group.

Next, we shall show that the dimension of the center of $\mathcal{G}$ is larger than or equal to that of the center of $\mathcal{U}/\mathcal{B}_0$. We regard $\mathcal{U}/\mathcal{B}_0$ as the structure group of the fibre bundle $(\mathcal{G}/\mathcal{B}_0, \tilde{p}, \mathcal{G}/\mathcal{U})$. Then, the associated principal bundle is $(\mathcal{G}/\mathcal{B}_0, \tilde{q}, \mathcal{G}/\mathcal{U})$, where $\tilde{q}$ denotes the canonical projection from $\mathcal{G}/\mathcal{B}_0$ onto $\mathcal{G}/\mathcal{U}$. Let $\bar{a}$ be a coset in $\mathcal{U}/\mathcal{B}_0$, and let $u$ be a representative of the coset $\bar{a}$. The holomorphic homeomorphism defined by $g \cdot \mathcal{B}_0 \to gu \cdot \mathcal{B}_0$, $g \in \mathcal{G}$, is determined by the coset $\bar{a}$, and is the right translation of the principal bundle $\mathcal{G}/\mathcal{B}_0$ corresponding to the element $\bar{a}$ of the structure group. Evidently, the right translation commutes with the mapping $g \cdot \mathcal{B}_0 \to xg \cdot \mathcal{B}_0$, $g \in \mathcal{G}$, assigned to an element $x$ in $\mathcal{G}$. We denote by $Z$ the identity connected component of the center of $\mathcal{U}/\mathcal{B}_0$, and by $n$ the complex dimension of $Z$. The complex Lie group $Z$, being a Lie subgroup in the structure group, acts on $\mathcal{G}/\mathcal{B}_0$ as a holomorphic transformation group. Let $X_1, \ldots, X_n$ be linearly independent holomorphic vector fields induced by one-parameter subgroups in $Z$. Then, at each point, they are linearly independent and form a base of the complex tangent space of the orbit of $Z$ through the point. Moreover, each $X_i$ is invariant by $\mathcal{G}$.

Denoting by $s$ the canonical projection of $\mathcal{G}/\mathcal{B}_0$ onto $\mathcal{G}/\mathcal{B}$, we obtain a holomorphic principal fibre bundle $(\mathcal{G}/\mathcal{B}_0, s, \mathcal{G}/\mathcal{B})$ whose structure group is the discrete subgroup $\mathcal{B}/\mathcal{B}_0$ in $\mathcal{U}/\mathcal{B}_0$. Since $Z$ is in the center of $\mathcal{U}/\mathcal{B}_0$, each of the holomorphic vector fields $X_1, \ldots, X_n$ is invariant by the action of the structure group $\mathcal{B}/\mathcal{B}_0$, and hence they are projectable. Let $Y_1, \ldots, Y_n$ be their image by the projection $s$. As $s$ is a local homeomorphism, $Y_1, \ldots, Y_n$ are linearly independent at each point. It is also obvious that each $Y_i$ is invariant by $\mathcal{G}$. The complex manifold $\mathcal{G}/\mathcal{B}$ being compact, the holomorphic vector fields $Y_1, \ldots, Y_n$ generate a connected complex Lie subgroup $\mathcal{Z}$ of dimension $n$ in $A_0(M)$. As $A_0(M) = \mathcal{G}$, $\mathcal{Z}$ is in the center of $\mathcal{G}$ and evidently in the radical $\mathcal{R}$ of $\mathcal{G}$.

In order to complete the proof, it suffices to show that the radical $\mathcal{R}$ of $\mathcal{G}$ is contained in the center of $\mathcal{G}$. Since $\mathcal{G}/\mathcal{U}$ is a Kaehler C-space, $\mathcal{R}$ is contained in $\mathcal{U}$, and so is $\mathcal{Z}$. First, we shall see that the image of $\mathcal{Z} \cdot \mathcal{B}$ under the canonical homomorphism $\tau: \mathcal{U} \to \mathcal{U}/\mathcal{B}_0$ contains $\mathcal{Z}$. For this purpose, we recall how the group $\mathcal{Z}$ is constructed. Take an
element $z$ in $\tau^{-1}(Z)$. To the right translation $f_{r(\sigma)}: g \cdot B_0 \to g\cdot z \cdot B_0, g \in G$, of the principal bundle $G/B_0$, there corresponds a holomorphic homeomorphism $g_{r(\sigma)}$ of $G/B$ onto itself, such that $s \cdot f_{r(\sigma)} = g_{r(\sigma)} \cdot s$. Hence, $g_{r(\sigma)}$ is the mapping $g \cdot B \to g\cdot z \cdot B$, $g \in G$. On the other hand, $g_{r(\sigma)}$ is realized by a mapping $g \cdot B \to z' \cdot g \cdot B, g \in G$, for a certain element $z'$ in $Z$. Therefore, we have $z \in z' \cdot B$, and $Z \subset \tau(\mathcal{Z} \cdot B)$. Since the image of $K$ under $\tau$ is in $Z$, we have $K \subset \mathcal{Z} \cdot B$. It follows that the orbit of $K$ through a point is equal to the orbit of $Z$; in fact, $K \cdot g \cdot B = g \cdot K \cdot B = g \cdot \mathcal{Z} \cdot B$ for any $g \in G$. Let $Y$ be a holomorphic vector field induced by a one-parameter subgroup in $K$. Then, $Y$ is expressed as a linear combination of $Y_1, \ldots, Y_n$ whose coefficients are holomorphic functions on $G/B$. Since $G/B$ is compact, all the coefficients must be constant. Thus, we have seen that the radical $K$ coincides with the central subgroup $Z$, and accordingly the dimension of the center of $U/B_0$ is equal to that of the center in $G$. From these facts, the assertion of Theorem 2 follows immediately.

Remark. As an immediate implication of the above theorem, we see that if $M$ is a Kaehler C-space, then $A_0(M)$ is semi-simple. This is a corollary of a theorem obtained by Matsushima (see Nagoya Math. J. 11). Indeed, $M$ is a complex coset space of a connected complex semi-simple Lie group $G$ by a closed connected complex Lie subgroup $B$, whose normalizer coincides with itself [3, (5.2)]. We may assume that $G$ is a subgroup of $A_0(M)$. From what we have shown in the above proof, it follows that $U/B_0$ reduces to the identity and accordingly so does the identity connected component of the center of the complex reductive Lie group $A_0(M)$. Thus, $A_0(M)$ is semi-simple.

Bibliography


Washington University