A THEOREM ON GENERIC NORMS OF STRICTLY POWER ASSOCIATIVE ALGEBRAS

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Let $A$ be a finite-dimensional strictly power associative algebra with an identity element over an arbitrary field $k$, and let

$$m(x) = x^n + \sum_{i=1}^{n} \lambda_i(x)x^{n-i}$$

be its generic minimum polynomial [1]. The coefficients $\lambda_i$ are polynomial functions on $A$. Such a function $f$ is called a Lie invariant under a linear transformation $L$ of the underlying vector space if $f(a+tL(a)) = f(a) \pmod{t^2}$ where $t$ is an indeterminate and $f$ is extended in the usual way to the vector space over $k(t)$; in particular, if $f$ is a linear form on $A$ (for instance the generic trace $\lambda_1$), this means that $f(L(a)) = 0$.

**Theorem.** The coefficients $\lambda_i$ of the generic minimum polynomial are Lie invariant under every derivation $d$ of $A$.

Assuming that $A$ is a Jordan algebra (over a field of characteristic not two), that $i = 1$ and that $d$ is the inner derivation which sends $a$ into $b \cdot ac - ba \cdot c$, we have the

**Corollary.** The identity $\lambda_1(b \cdot ac) = \lambda_1(ba \cdot c)$ holds in any Jordan algebra.

This result has been obtained independently by N. Jacobson (unpublished).

**Proof of the theorem.** Let $K$ be an arbitrary extension of $k$. The extensions of the forms $\lambda_i$ and of the derivation $d$ to $A_K$ will be denoted by the same symbols $\lambda_i$ and $d$. Let $t$ be an indeterminate scalar and, for $a, b \in A_K$, denote by $\{a, b\}_i$ (resp. $\mu_i(a, b)$) the coefficient of $t$ in $(a+tb)^i$ (resp. in $\lambda_i(a+tb)$). As $m(a+tb)$ vanishes identically, the coefficient of $t$ in it must be zero, that is,

$$\{a, b\}_n + \sum_{i=1}^{n} \lambda_i(a) \cdot \{a, b\}_{n-i} + \sum_{i=1}^{n} \mu_i(a, b) \cdot a^{n-i} = 0 \quad (1)$$

It is easily seen that $d(a^i) = \{a, d(a)\}_i$; therefore

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\[ d(m(a)) = \{ a, d(a) \} \cdot \sum_{i=1}^{n} \lambda_i(a) \cdot \{ a, d(a) \} \cdot a^{-i} = 0 \]

for every \( a \in A_K \). Setting \( b = d(a) \) in (1) and subtracting (2), we have

\[ \sum_{i=1}^{n} \mu_i(a, d(a)) \cdot a^{-i} = 0. \]

If \( a \) is generic (over \( k \)), it does not satisfy any polynomial identity of order \( n - 1 \), with coefficients in \( K \); thus

\[ \mu_i(a, d(a)) = 0, \]

and the same relations then hold for arbitrary \( a \in A_K \).

By the definition of the \( \mu_i(a, b) \) this is the Lie invariance of the \( \lambda_i(a) \) which we wished to prove.

**Bibliography**


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