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**COMPLETELY WELL-POSED PROBLEMS FOR NONLINEAR DIFFERENTIAL EQUATIONS**

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1. Well-posed and completely well-posed problems for linear partial differential equations have been discussed by Hormander [2] and more recently and more generally by Browder [1]. Roughly speaking, if $L$ is a differential operator in a Banach space $X$, the problem of finding a solution of $Lu = f$, $f \in X$, is said to be (completely) well-posed if the range of $L$ is $X$ and if in addition $L^{-1}$ exists and is (completely) continuous. In both papers, sufficient conditions are given for the existence of well-posed and completely well-posed problems for formal differential operators.

In this paper we are interested in the effect on a completely well-posed problem of a nonlinear perturbation of the operator $L$. In particular, we will show (Theorem 3) that under certain conditions a completely well-posed problem for a differential operator $L$ remains completely well-posed for $L + A$, where $A$ is a nonlinear transformation in $X$. Combining this result with theorems in [1], conditions guaranteeing the existence of completely well-posed problems for perturbed differential operators can be derived. One such result is given in Theorem 4 for the case $X = L^2$.

2. Let $X$ be a Banach space, $T$ a transformation with domain $D(T) \subset X$ and range $R(T) \subset X$. The transformations here are not assumed to be linear unless it is so stated.

¹ This research was supported by the Air Force Office of Scientific Research.
Definition. The transformation $T$ is said to be asymptotic to zero if $D(T) = X$ and
\[
\lim_{\|u\| \to \infty} \frac{\|Tu\|}{\|u\|} = 0.
\]
This definition is due to Krasnoselskiï and the following theorem, referred to in [3], to Dubrovskii, who used it in treating nonlinear integral equations. A proof of this theorem is included here for the convenience of the reader.

Theorem 1. If $T$ is completely continuous and asymptotic to zero, then $R(I+T) = X$.

Proof. Let $f$ be an arbitrary element of $X$. To prove that $u + Tu = f$ has a solution, it suffices to prove that the transformation $S$ defined by $Su = f - Tu$ has a fixed point. Noting that $S$ is completely continuous since $T$ is, we need only show, using the Schauder theorem, that $S$ maps some closed sphere of $X$ into itself.

Let
\[
B_r(f) = \{ v \in X \mid \|v - f\| \leq r \}
\]
and suppose that for each integer $n > 0$, the set $SB_n(f)$ contains an element $Su_n$ not in $B_n(f)$. Then the sequence $\{u_n\}$ has the property that $\|u_n - f\| \leq n$, while $\|Su_n - f\| = \|Tu_n\| > n$. Since $T$ is completely continuous and the sequence $\{Tu_n\}$ is unbounded, $\{u_n\}$ is also unbounded. On the other hand $\|u_n\| \leq \|f\| + n$, so that
\[
\frac{\|Tu_n\|}{\|u_n\|} > \frac{n}{\|f\| + n}.
\]
But this contradicts the assumption that $T$ is asymptotic to zero, since for such an operator $\|Tu\|/\|u\|$ cannot be bounded away from zero as $u$ ranges over an unbounded set. Consequently, $S$ maps some $B_n(f)$ into itself, completing the proof.

The main result of this section is:

Theorem 2. Let $L$ be a linear transformation, not necessarily bounded, with domain $D(L) \subseteq X$ and $R(L) = X$, and suppose $L$ has a completely continuous inverse. Let $A$ be bounded, continuous, and asymptotic to zero. Then $R(L + A) = X$.

The proof will follow easily from Theorem 1 with the aid of the following lemma.

Lemma. If $D(A) = D(K) = X$, and $K$ is linear and bounded while $A$
is bounded and asymptotic to zero, then \( AK \) is asymptotic to zero.

**Proof.** It will be convenient to introduce some additional notation. Since \( A \) is bounded, there is for each \( r \geq 0 \) a number \( s \geq 0 \) such that \( AB_r(0) \subset B_s(0) \), using the notation introduced in (1). Denote by \( M(r) \) the greatest lower bound of the set of such \( s \). Also, since \( A \) is asymptotic to zero, there is a real non-negative function \( P(e) \) defined for \( e > 0 \) such that \( \|Av\| < e\|v\| \) whenever \( \|v\| > P(e) \). Given \( e > 0 \), set \( \rho = P(e/\|K\|) \) (if \( K = 0 \) the lemma is trivial) and let \( u \) be any element of \( X \) such that \( e\|u\| > M(\rho) \). There are two cases to consider, according as \( \|Ku\| > \rho \) or \( \|Ku\| \leq \rho \). In the first case,

\[
\frac{\|AKu\|}{\|u\|} = \frac{\|AKu\|}{\|Ku\|} \frac{\|Ku\|}{\|u\|} \leq \frac{\|AKu\|}{\|Ku\|} \frac{\|K\|}{\|K\|} < \frac{e}{\|K\|} \|K\| = e
\]

while in the second case

\[
\frac{\|AKu\|}{\|u\|} < \frac{M(\rho)}{\|K\|} \leq \frac{\|K\|}{\|K\|} = e
\]

so that in any event, \( \|AKu\| < e\|u\| \) for \( \|u\| \) sufficiently large.

**Proof of Theorem 2.** Since \( D(L + A) = D(L) \), \( (L + A)L^{-1} = I + AL^{-1} = I + T \) is everywhere defined and from the lemma, \( T \) is asymptotic to zero. Furthermore, since \( A \) is continuous and \( L^{-1} \) completely continuous, \( T \) is completely continuous. Thus \( R(I + T) = X \) by Theorem 1 and since \( R(L) = X \) by hypothesis, it follows that \( R(L + A) = R((I + T)L) = X \).

It should perhaps be noted here that Theorem 1 remains true under weaker hypotheses. For example, as is clear from the proof, one need only assume that for \( \|u\| \) sufficiently large, \( \|Tu\| \leq c\|u\| \) for some \( c < 1 \). However, if \( A \) has only this weaker property, \( T = AL^{-1} \) need not have, unless some restriction is made on \( \|L^{-1}\| \).

3. Let \( K \) be any transformation in \( X \). The problem of finding a solution \( u \in D(K) \) of \( Ku = f, f \in X \), is said to be (completely) well-posed if \( R(K) = X \) and \( K \) has a (completely) continuous inverse. In this section we wish to consider the case in which \( K \) is a differential operator and \( X \) is a complex \( L^p \) space, \( 1 \leq p < \infty \). The description which follows is admittedly brief; full details can be found in [1].

Let \( G \) be a bounded, open subset of Euclidean \( n \)-space, \( n \geq 1 \). In now standard notation, we denote by

\[
P = \sum_{|\alpha| \leq m} a_\alpha(x) D^\alpha
\]

a linear differential operator of order \( m \), with coefficients \( a_\alpha(x) \) com-
plex-valued functions on \( G \), and by \( P' \), defined by

\[
P' u = \sum_{|\alpha| \leq m} D^\alpha (\bar{a}_\alpha(x) u)
\]

its formal adjoint. We first consider \( P \) and \( P' \) defined on \( C_0^\infty(G) \), the infinitely-differentiable complex functions with compact support in \( G \), and then close them as operators in \( L^p(G) \) and \( L^{p'}(G) \) respectively, where \( p' = p/(p-1) \). These new operators will be denoted by \( P_0 \) and \( P_0^* \). \( P_0 \) and \( P_1 \), the restricted adjoint of \( P_0^* \), are called the minimal and maximal operators associated with the formal differential operator \( P \). Let \( L \) be a closed linear operator with \( P_0 \subseteq L \subseteq P_1 \). If \( R(L) = L^p(G) \) and \( L \) has a (completely) continuous inverse, then \( L \) is said to be a (completely) solvable realization of the pair \( (P_0, P_0^*) \) and the problem \( Lu = f \) is then (completely) well-posed.

**Theorem 3.** Let \( g(x, z) \) be a complex-valued function defined and uniformly continuous for \( x \in G \), all complex \( z \), such that

(2) \[
|g(x, z)| \leq c_1 |z|^a + c_2,
\]

where \( c_1, c_2, \) and \( a \) are non-negative constants and \( a < 1 \). Denote by \( A \) the operator in \( L^p(G) \) defined by \( Au(x) = g(x, u(x)) \). If \( L \) is a completely solvable realization of the pair \( (P_0, P_0^*) \), then the problem \( (L+A)u = f \), \( f \in L^p(G) \), is completely well-posed provided \( L+A \) has a continuous inverse.

**Proof.** The conditions imposed on \( g(x, z) \) insure that \( A \) is a bounded and continuous operator defined on \( L^p(G) \). (Actually, less will suffice to give the same result. References to papers giving results along this line can be found in [3].) That \( A \) is asymptotic to zero follows easily from (2) and hence \( R(L+A) = L^p(G) \) from Theorem 2. Since \( L+A \) has a continuous inverse by hypothesis, it remains only to show that in fact \( (L+A)^{-1} \) is completely continuous. In view of the complete continuity of \( L^{-1} \), it suffices to show that if \( \{u_n\} \) is any sequence of elements in \( D(L) = D(L+A) \) such that the set \( \{ (L+A)u_n \} \) is bounded, then the set \( \{Lu_n\} \) is also bounded. Suppose then that \( \| (L+A)u_n \| < M \) while \( \{Lu_n\} \) is unbounded. Eliminating those \( n \)'s for which \( Lu_n = 0 \) and setting \( v_n = Lu_n \) for the remainder, we have

\[
\|AL^{-1}v_n\| = \|Au_n\| \geq \|Lu_n\| - M = \|v_n\| - M,
\]

so that

\[
\frac{\|AL^{-1}v_n\|}{\|v_n\|} \geq 1 - \frac{M}{\|v_n\|}.
\]

But, as in the proof of Theorem 1, this last inequality is impossible.
since $AL^{-1}$ is asymptotic to zero. Thus $\{Lu_n\}$ is a bounded set and so $\{u_n\}$ has a convergent subsequence.

It is clear that the theorem could be immediately generalized to transformations of the form $L+KAM$, where $A$ is as described in the theorem and $K$ and $M$ are linear and bounded. That $KAM$ is asymptotic to zero follows easily from the lemma.

Finally, a word may be said about the existence of completely well-posed problems for the formal differential operator $P+A$. By combining these results with theorems of [1] and [2], a number of results can be obtained. One example will suffice to illustrate the idea.

**Theorem 4.** Let $p = 2$ and let $Au = g(x, u(x))$, where $g(x, z)$ satisfies the conditions of Theorem 3. Suppose $P_0$ and $P_0^*$ have completely continuous inverses on their respective ranges. Then there exists a completely solvable realization $L$ of the pair $(P_0, P_0^*)$ and a number $c > 0$ such that if $|g(x, u) - g(x, v)| \leq c|u - v|$ for all complex $u, v$ and all $x$ in $G$, then the problem $(L+A)u = f, f \in L^2(G)$, is completely well-posed.

**Proof.** The existence of $L$ follows from Theorem 1.2 of [2]. (The corresponding theorem for reflexive Banach spaces can be found in [1].) An easy calculation shows that if $|g(x, u) - g(x, v)| \leq c|u - v|$, where $c||L^{-1}|| < 1$, then for $u(x), v(x) \in D(L) = D(L+A),

$$\|u - v\| \leq ||L^{-1}||^2(1 - c||L^{-1}||)^{-1}||L + A||u - (L + A)v||.$$

Hence $L+A$ has a continuous inverse and the conclusion follows from Theorem 3.

Note that the constant $c$ can be estimated from the construction of $L^{-1}$ given in the proof of Theorem 1.2 in [2]. Hörmander also gives in [2] conditions which insure the complete continuity of $P_0^{-1}$ and $(P_0^*)^{-1}$.

**References**


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