RELATIONS AMONG STIEFEL WHITNEY CLASSES

ROBERT E. STONG

The object of this paper is to consider the problems: What are the relations among the Whitney classes of all manifolds of dimension \( n \)? In other words, which \( a \in H^*(BO, \mathbb{Z}_2) \) have the property that \( a(M^n) = 0 \) for all \( n \)-dimensional manifolds.

For a connected manifold \( M^n \), we denote by \( I_M \) the kernel of the homomorphism \( r^*: H^*(BO, \mathbb{Z}_2) \to H^*(M^n, \mathbb{Z}_2) \) induced by the classifying map for the tangent bundle of \( M^n \).

If \( M \) and \( N \) are connected \( n \)-dimensional manifolds, and if we form the connected sum \( [2] \) of \( M \) and \( N \), which will be denoted \( M \# N \), then in dimensions between 0 and \( n \), the cohomology of \( M \# N \) can be decomposed into the direct sum of the cohomology of \( M \) and that of \( N \). Further, this decomposition is compatible with the maps from \( H^*(BO, \mathbb{Z}_2) \). Thus, in dimensions between 0 and \( n \), \( I_M \cap I_N = I_M \cap I_N \).

To decrease the ideal associated with a manifold, it is then reasonable to form the connected sum with another manifold. This process will be used to show:

**Theorem.** There is no relation of dimension less than or equal to the integral part of \( n/2 \) among the Whitney classes of all manifolds of dimension \( n \).

This result has also been obtained by an entirely different method by E. H. Brown, Jr. [1].

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**Proof of the Theorem.** Let \( M_{2k} = \#_{(a_1, \ldots, a_p) \in \Sigma(k)} \text{RP}_{2a_1} \times \cdots \times \text{RP}_{2a_p} \) be the connected sum over all elements of the set of partitions of the integer \( k \) of the products of even dimensional real projective spaces.

(1) It suffices to show that for each \( k \), \( r^*: H^k(BO, \mathbb{Z}_2) \to H^k(M_{2k}, \mathbb{Z}_2) \) is a monomorphism.

Let \( a \in H^m(BO, \mathbb{Z}_2) \) with \( a(M) = 0 \) for all manifolds \( M \) of dimension \( n \), and suppose \( m \leq \lfloor n/2 \rfloor \). If \( n = 2k \), let \( M^n = M_{2k} \); if \( n = 2k+1 \), let \( M^n = M_{2k} \times S^1 \). Then \( m \leq k \), and so \( r^*_M(a \cdot w_{k-m}) = r^*_M(a) \cdot r^*_M(w_{k-m}) = 0 \).

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If \( n = 2k \), this makes \( \alpha \cdot w_{2m} = 0 \), for \( \tau^* \) is assumed monic. If \( n = 2k + 1 \),
\( w(M_{2k} \times S^1) = w(M_{2k}) \otimes w(S^1) = w(M_{2k}) \otimes 1 \) in \( H^* (M_{2k} \times S^1, \mathbb{Z}_2) \), so
\( \tau^* (\alpha \cdot w_{2m}) = \tau^* (\alpha \cdot w_{2m}) \otimes 1 \). Since \( \tau^* \) is assumed monic, this also gives \( \alpha \cdot w_{2m} = 0 \). Since \( H^* (BO, \mathbb{Z}_2) \) is a polynomial algebra over \( \mathbb{Z}_2 \), it has no divisors of zero and hence \( \alpha = 0 \).

(2) Let \( \overline{M}_{2k} \) denote the disjoint union \( \bigcup_{(a_1, \ldots, a_r) \in \pi(k)} RP_{a_1} \times \cdots \times RP_{a_r} \). The cohomology of \( \overline{M}_{2k} \) is the direct sum of the algebras \( H^* (RP_{a_1} \times \cdots \times RP_{a_r}, \mathbb{Z}_2) \) whose structure is well known. The Mayer-Vietoris sequence shows that \( H^k (M_{2k}, \mathbb{Z}_2) \) is isomorphic to \( H^k (\overline{M}_{2k}, \mathbb{Z}_2) \) under an isomorphism \( \phi \) such that
\[
H^k (BO, \mathbb{Z}_2) \xrightarrow{\tau^*} H^k (M_{2k}, \mathbb{Z}_2) \xrightarrow{\phi} H^k (\overline{M}_{2k}, \mathbb{Z}_2)
\]
commutes.

Define a homomorphism \( \Omega : H^k (BO, \mathbb{Z}_2) \to H^{2k} (\overline{M}_{2k}, \mathbb{Z}_2) \) by
\[
H^k (BO, \mathbb{Z}_2) \xrightarrow{\tilde{\tau}^*} H^k (\overline{M}_{2k}, \mathbb{Z}_2) \xrightarrow{S^k} H^{2k} (BO, \mathbb{Z}_2) \xrightarrow{\phi} H^{2k} (\overline{M}_{2k}, \mathbb{Z}_2).
\]
Since \( \Omega = S^k \cdot \tilde{\tau}^* = S^k \cdot \phi \cdot \tau^* \), it will suffice to show that \( \Omega \) is an isomorphism.

(3) Consider the elements \( w_i \in H^* (BO, \mathbb{Z}_2) \) as the \( i \)th elementary symmetric functions \( \sigma_i \) of variables \( t_a \). For each partition \( (i_1, \ldots, i_r) \) in \( \pi(k) \), we define \( s_{(i_1, \ldots, i_r)} (\sigma_1, \ldots, \sigma_k) \) to be the symmetric function \( \sum i_1^{i_1} \cdots i_r^{i_r} \). As is well known, the functions \( s_{\omega}(\sigma_1, \ldots, \sigma_k) \) for \( \omega \in \pi(k) \) form an additive basis for the symmetric functions in the \( i \)'s which are homogeneous of degree \( k \). Thus the elements \( s_{\omega}(w_1, \ldots, w_k) \) for \( \omega \in \pi(k) \) form an additive basis of \( H^k (BO, \mathbb{Z}_2) \). (See [3].)

For each \( \omega = (i_1, \ldots, i_r) \in \pi(k) \), let \( 2\omega = (2i_1, \ldots, 2i_r) \in \pi(2k) \), and let \( RP_{2\omega} = RP_{i_1} \times \cdots \times RP_{i_r} \). Considering \( H^{2k} (\overline{M}_{2k}, \mathbb{Z}_2) \) as \( \bigoplus H^{2k} (RP_{2\omega}, \mathbb{Z}_2) \), it will then suffice to show that the matrix
\[
\| S^k \cdot \tilde{\tau}^* s_{\omega} (\text{RP}_{2\omega}) \|_{\omega \in \pi(k)}
\]
over \( \mathbb{Z}_2 \) is nonsingular, in order to show that \( \Omega \) is an isomorphism.

(4) Since \( S^k \tilde{\tau}^* = \tilde{\tau}^* S^k \), we consider the elements \( S^k s_{\omega} \in H^{2k} (BO, \mathbb{Z}_2) \). Since \( s_{\omega} \in H^k (BO, \mathbb{Z}_2) \),
\[
S^k s_{\omega} = S^k s_{(i_1, \ldots, i_r)} = (s_{(i_1, \ldots, i_r)})^2 = (\sum i_1^{i_1} \cdots i_r^{i_r})^2
\]
\[
= \sum i_1^{i_1} \cdots i_r^{i_r} = s_{\omega}.
\]
Thus the matrix $\|s^*\tau^*w(RP_{2n})\|_{\omega, \mu \in \pi(2k)}$ can be considered as a sub-matrix of

$$\|\tau^*s_\mu(RP_\lambda)\|_{\lambda, \mu \in \pi(2k)}.$$ 

Now the element

$$\tau^*s_\mu(RP_\lambda) = \tau^*s_\mu(RP_{b_1} \times \cdots \times RP_{b_l}),$$

$$= \sum_{\mu_1, \cdots, \mu_l = \mu} \tau^*s_{\mu_1}(RP_{b_1}) \otimes \cdots \otimes \tau^*s_{\mu_l}(RP_{b_l}),$$

and so $\tau^*s_\mu(RP_\lambda) = 0$ unless $\mu$ refines $\lambda$. (See [3].) Putting a total order on the set $\pi(2k)$, compatible with the relation $\omega \geq \omega$ if $\omega$ is a refinement of $\omega$, makes the matrix $\|\tau^*s_\mu(RP_\lambda)\|$ a triangular matrix. Hence also $\|s^k\tau^*w(RP_{2n})\|$ is triangular and has diagonal elements

$$\{\tau^*s_{2n}(RP_{2n})\}.$$

If $\omega = (\alpha_1, \cdots, \alpha_r)$, this becomes

$$\tau^*s_{2n}(RP_{2n}) = \tau^*s_{2n_1}(RP_{2n_1}) \otimes \cdots \otimes \tau^*s_{2n_r}(RP_{2n_r}).$$

Since $w(RP_{2n_\alpha}) = (1 + \alpha)^{2n_\alpha}$, for $\alpha \in H^1(RP_{2n_\alpha}, Z_2)$, $w_\alpha(RP_{2n_\alpha})$ can be considered as the $i$th elementary symmetric function in the $2\alpha + 1$ variables $\alpha, \cdots, \alpha$. Thus

$$\tau^*s_{2n_\alpha}(RP_{2n_\alpha}) = \sum (\alpha)^{2n_\alpha},$$

$$= (2\alpha + 1)\alpha^{2n_\alpha},$$

$$= \alpha^{2n_\alpha},$$

and so

$$\tau^*s_{2n}(RP_{2n}) = \alpha_1 \otimes \cdots \otimes \alpha_r,$$

which when considered as a Whitney number is 1.

This means that the matrix $\|s^k\tau^*w(RP_{2n})\|$ is triangular, with all diagonal entries having value 1, and hence this makes the matrix nonsingular, completing the proof of the theorem.

REFERENCES

2. J. Milnor, *Differentiable manifolds which are homotopy spheres* (mimeographed), Princeton University, Princeton, N. J.

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