AN EXAMPLE FOR HOMOTOPY GROUPS WITH COEFFICIENTS

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In this note we present two spaces $X$ and $Y$ all of whose homotopy groups are isomorphic, but whose homotopy groups with coefficients are not isomorphic for a certain coefficient group.¹ This example depends on the fact that the universal coefficient sequence [2], which relates the ordinary homotopy groups to those with coefficients, does not split. The spaces $X$ and $Y$ will be 1-connected, CW-complexes and the coefficient group will be $\mathbb{Z}_m$, the integers modulo $m$.

We adopt the following notation: $M(G, p)$ denotes a Moore complex of type $(G, p)$ (i.e., a space with a single nonvanishing homology group $G$ in dimension $p$) and $K(G, p)$ denotes an Eilenberg-MacLane complex of type $(G, p)$ (i.e., a space with a single nonvanishing homotopy group $G$ in dimension $p$). Recall that $\pi_r(G; A)$, the $r$th homotopy group of the space $A$ with coefficients in the group $G$, is the group of homotopy classes of base point preserving maps from $M(G, r)$ into $A$. If $\mathbb{Z}$ is the group of integers, then $\pi_r(\mathbb{Z}; A) = \pi_r(A)$, the $r$th homotopy group of $A$. Finally we recall the universal coefficient theorem [2] which asserts the exactness of the following sequence

$$0 \to \text{Ext}(G, \pi_{r+1}(A)) \to \pi_r(G; A) \to \text{Hom}(G, \pi_r(A)) \to 0.$$  

Now let $X = M(\mathbb{Z}_m, r)$ and let $d$ be the greatest common divisor of $m$ and $n$. We shall always assume that $d$ is even, that $8$ does not divide $mn$, and that $r$ is an integer $>2$. Under these conditions Barratt [1] has shown that $\pi_r(\mathbb{Z}_m; X) \cong \mathbb{Z}_{2d}$. In addition, it is well known that $\pi_{r+1}(X) \cong \mathbb{Z}_n \otimes \mathbb{Z}_2 = \mathbb{Z}_2$. Now let $X$ be the space obtained from $X$ by attaching cells to kill all homotopy groups in dimensions $\geq r+2$. Thus

$$\pi_i(X) = 0 \text{ for all } i \leq r - 1 \text{ and } \geq r + 2,$$

$$\pi_r(X) = \mathbb{Z}_n \text{ and } \pi_{r+1}(X) = \mathbb{Z}_2.$$  

Furthermore, since $M(\mathbb{Z}_m, r)$ is an $(r+1)$-dimensional CW-complex and the $(r+2)$-skeleton of $X$ is $X$, it follows from standard cellular approximation arguments that $\pi_r(\mathbb{Z}_m; X) \approx \pi_r(\mathbb{Z}_m; \overline{X})$. Hence we have

$$\pi_r(\mathbb{Z}_m; X) \approx \mathbb{Z}_{2d}.$$  

Next let $Y$ be a product of Eilenberg-MacLane spaces, $Y = K(\mathbb{Z}_n, r)$

¹ We have been informed that such an example was known to Eckmann and Hilton.
\( \times K(Z_2, r+1) \). Since \( \pi_i(Y) = \pi_i(K(Z_n, r)) \oplus \pi_i(K(Z_2, r+1)) \), by (2) we have that \( \pi_i(X) \) and \( \pi_i(Y) \) are isomorphic for all \( i \). Now we consider \( \pi_r(Z_m; Y) \). Clearly \( \pi_r(Z_m; Y) = \pi_r(Z_m; K(Z_n, r)) \oplus \pi_r(Z_m; K(Z_2, r+1)) \).

By (1), \( \pi_r(Z_m; K(Z_n, r)) = \text{Hom}(Z_m, Z_n) = Z_d \). Also by (1),

\[
\pi_r(Z_m; K(Z_2, r+1)) = \text{Ext}(Z_m, Z_2) = Z_2.
\]

Thus

\[
(4) \quad \pi_r(Z_m; Y) \approx Z_d \oplus Z_2.
\]

A comparison of (3) and (4) shows that \( \pi_r(Z_m; X) \) is not isomorphic to \( \pi_r(Z_m; Y) \). But we have already seen that \( \pi_i(X) \approx \pi_i(Y) \) for all \( i \). This completes the example.

We observe that the spaces \( X \) and \( Y \) may be distinguished by invariants other than the \( r \)th homotopy group with coefficients in \( Z_m \). For instance, it is easily seen that \( H_{r+1}(X) = 0 \) and \( H_{r+1}(Y) = Z_2 \).

In closing we note that the spaces \( X \) and \( Y \) serve as an example for homotopy groups with coefficients as defined by Katuta [3]. This is so since, for a finite coefficient group, Katuta's groups are the same as the ones we consider, except for a dimensional shift of one unit.

References


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