SOME ORTHOGONAL FUNCTIONS CONNECTED
WITH POLYNOMIAL IDENTITIES. II

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In an earlier paper [5] we proved a general polynomial identity, some special cases of which gave rise to sets of orthogonal functions
[6]. In this paper we recast this general polynomial identity into
such a form that it leads directly to a general class of orthogonal functions
containing, as special cases, those given in [6] and also those
introduced in [4].

Let \( n_1, n_2, \ldots \) be a sequence of integers each \( \geq 2 \) and let \( p_0 = 1 \)
and \( p_j = n_1 n_2 \ldots n_j \) for \( j \geq 1 \). Then each integer \( n, 0 \leq n < p_m \), may
be written uniquely in the form
\[
(1) \quad n = a_0 + a_1 p_1 + \cdots + a_{m-1} p_{m-1}, \quad 0 \leq a_j < n_{j+1},
\]
and we have
\[
(2) \quad a_j = [n/p_j] - n_{j+1}[n/p_{j+1}].
\]

In [5] we proved the following: if \( \sum_{n=0}^{p_j-1} f_j(n)n^t = 0 \) for \( 0 \leq t \leq \alpha_j \)
and \( P \) is any polynomial of degree \( \leq \alpha_1 + \cdots + \alpha_m + m - 1 \) then
\[
(3) \quad \sum_{n=0}^{p_m-1} \prod_{j=1}^{m} f_j([n/p_{j-1}] - n_j [n/p_j]) P(x + n) = 0.
\]

Since (3) holds for any integers \( n_1, \ldots, n_m \) (each \( \geq 2 \)) it holds for
\( n'_1, \ldots, n'_m \), where \( n'_j = n_{m-j+1} \). Putting \( p'_j = n'_1 \cdots n'_j \) and writing \( f'_j = f_{m-j+1} \) we see that the conditions leading to (3) become
conditions leading to
\[
\sum_{n=0}^{p'_m-1} \prod_{j=1}^{m} f'_j([n/p'_j-1] - n'_j [n/p'_j]) P(x + n) = 0
\]
for the same polynomials \( P \). Noting that \( p'_j = p_m/p_{m-j} \) we have the
following: if \( \sum_{n=0}^{p'_j-1} f'_j(n)n^t = 0 \) for \( 0 \leq t \leq \alpha_j \) and \( P \) is any polynomial
of degree \( \leq \alpha_1 + \cdots + \alpha_m + m - 1 \) then
\[
(4) \quad \sum_{n=0}^{p'_m-1} \prod_{j=1}^{m} f'_j([np_j/p_m] - n_j [np_{j-1}/p_m]) P(x + n) = 0.
\]

There are \( p_m \) coefficients in this identity and we define \( G_m(x) \) to
be the periodic step function, with period unity, taking the value of

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the $i$th coefficient in (4) on the $i$th subinterval of length $1/p_d$ in $[0, 1)$, it is easy to see that

$$G_m(x) = \prod_{j=1}^{m} f_j([p_jx] - n_j[p_{j-1}x]).$$

Such a function is defined for each natural number $m$.

For convenience in our future expressions we introduce two other sequences of functions of $x$ suggested by (5).

$$v_j(x) = [p_jx] - n_j[p_{j-1}x] \quad \text{for } j = 1, 2, \ldots.$$  

$$\phi_{j-1}(x) = f_j(v_j(x))$$

It is clear that $G_m(x) = \prod_{j=0}^{m-1} \phi_j(x)$ and

$$v_j(x + 1/p_{j-1}) = v_j(x), \quad j = 1, 2, \ldots.$$  

The functions $v_j(x)$ have a more obvious interpretation than that given in the above context. Indeed, these functions are merely the digits in the Cantor expansion (see [3, p. 7]) of $x - [x]$ relative to the $p_j$.

$$x - [x] = v_1(x)/p_1 + v_2(x)/p_2 + \cdots.$$  

Writing $\mu\{v_1 \leq d_1, \ldots, v_n \leq d_n\}$ for the Lebesgue measure of the set of $x$, $0 \leq x < 1$, for which $v_i(x) \leq d_i, \ldots, v_n(x) \leq d_n$ we see that

$$\mu\{v_1 \leq d_1, \ldots, v_m \leq d_m\} = \sum \mu\{v_1 = a_1, \ldots, v_m = a_m\} = (d_1 + 1)(d_2 + 1) \cdots (d_m + 1)/p_m,$$

where the sum is taken over all $m$-tuples $(a_1, \ldots, a_m)$ for which $0 \leq a_j \leq d_j$. Also

$$\mu\{v_j \leq d_j\} = \mu\{v_1 \leq n_1 - 1, \ldots, v_{j-1} \leq n_{j-1} - 1, v_j \leq d_j\}$$

$$= n_1 \cdots n_{j-1}(d_j + 1)/p_j$$

$$= (d_j + 1)/n_j.$$

Therefore

$$\mu\{v_1 \leq d_1, \ldots, v_m \leq d_m\} = \mu\{v_1 \leq d_1\} \cdots \mu\{v_m \leq d_m\};$$

i.e., the $v_j$ functions are statistically independent (see [2]).

Using (9) we give an easy proof of

$$\int_0^1 \prod_{j=0}^{n-1} \phi_j(x)dx = \prod_{j=0}^{n-1} \int_0^1 \phi_j(x)dx \left(= \prod_{j=0}^{n-1} \sum_{n=0}^{s-1} f_{j+1}(n) \right).$$

The second equality in (10) is clear and the first goes as follows.
\[ \int_0^1 \prod_{j=0}^{j-1} \phi_j(x) dx = \int_0^1 \prod_{j=1}^{j-1} (\nu_j(x)) dx \]
\[ = \sum_{a_1, \ldots, a_j \geq a_j < n_j} \prod_{j=1}^{j-1} f_j^{\beta_j-1}(a_j) \mu \{ \nu_1 = a_1, \ldots, \nu_s = a_s \} \]
\[ = \sum_{a_1, \ldots, a_j \geq a_j < n_j} \prod_{j=1}^{j-1} (f_j^{\beta_j-1}(a_j)) \mu \{ \nu_j = a_j \} \]
\[ = \prod_{j=1}^{j-1} \int_0^1 (\nu_j(x)) dx = \prod_{j=0}^{j-1} \int_0^1 \phi_j(x) dx. \]

When the \( f_j \) satisfy the conditions leading to (4) the identity (10) guarantees the vanishing of the integral of power products of the \( \phi_j \) functions in which at least one exponent \( \beta_j \) is unity. In particular the functions \( G_m(x) \) in (5) are orthogonal.

In the special case
\[ f_j(n) = (-1)^n \left( \begin{array}{c} n_j - 1 \\ n \end{array} \right) \]
equation (4) becomes
\[ \sum_{n=0}^{m-1} (-1)^{a_0 + a_{1-1} + \cdots + a_{m-1}} \left( \begin{array}{c} n_m - 1 \\ a_0 \\ a_{m-1} \end{array} \right) P(x + n) = 0 \]
for \( P \) a polynomial of degree \( \leq n_1 + \cdots + n_m - m - 1 \) and where, the \( a_i \) are defined by
\[ n = a_0 + a_1 p_m / p_{m-1} + a_2 p_m / p_{m-2} + \cdots + a_{m-1} p_m / p_1, \quad 0 \leq a_j < n_{m-j}. \]
The corresponding functions \( G_m \), given in (5), are orthogonal and in the case where all \( n_j = b \geq 2 \) are the functions \( t_m \) defined in \( \texttt{[6]} \). In this case also the \( s_n \) functions of that paper are the \( \phi_j \) functions here. Thus Theorem 3 of that paper, dealing with power products of the \( s_n \) functions, is contained in (10).

In the special case \( f_j(n) = e^{2\pi i n_j / n_j} \) we have
\[ \phi_{j-1}(x) = f_j(\nu_j(x)) = e^{2\pi [p_j x - n_j [p_{j-1} x]] / n_j} = e^{2\pi [p_j x] i / n_j}. \]

These are the orthogonal functions \( \phi_n \) defined in \( \texttt{[4]} \). Again (10) furnishes us with a proof of the orthogonality of power products of these functions. Indeed, if \( \Phi_n \) and \( \Phi_m \) are two such power products then \( \Phi_n \Phi_m \) is of the form \( \phi_0^{\beta_1} \cdots \phi_{k+1} \), where \( 0 \leq \beta_j < n_j \), and not all \( \beta_j = 0 \) when \( n \neq m \). We now need only observe that
\[ \sum_{n=0}^{n_j-1} (e^{2\pi i n_j / n_j})^{\beta_i} = 0 \quad \text{for } 0 < \beta_j < n_j. \]
In particular when all \( n_i \) are equal to \( b \) the \( \phi_j \) are the Rademacher and the generalized Rademacher functions [1] in the cases where \( b = 2 \) and \( b > 2 \), respectively.

References