ON THE POINT SPECTRUM OF POSITIVE OPERATORS

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1. Recently, G.-C. Rota proved the following result:

Let \((S, \Sigma, \mu)\) be a measure space of finite measure, \(P\) a positive
linear operator on \(L_1(S, \Sigma, \mu)\) with \(L_1\)-norm and \(L_\infty\)-norm at most one.
If \(\alpha, |\alpha| = 1\), is an eigenvalue of \(P\) such that \(\alpha f = Pf\) (\(f \in L_1\)), then \(\alpha^2\)
is an eigenvalue such that \(\alpha^2 |f|^2 = P(|f|^2 g^2)\), where \(g = |f| g\).

It can be added that \(\alpha^n |f|^n = P(|f|^n g^n)\) for every integer \(n\); thus
Rota proved for a fairly large class of operators, without compactness
assumptions, a result known (and due to Frobenius) for positive
finite square matrices, and known for certain types of positive oper-
ators under conditions guaranteeing that the spectrum intersects the
circumference of the spectral circle but in a finite set (see Karlin [1,
pp. 933–935] for an excellent survey and some more general exam-
pies). Simple but typical examples of operators showing the spectral
behavior exhibited in Rota’s theorem are the permutation matrices
on \(l_p\) (\(1 \leq p \leq \infty\)).

The purpose of this paper is to extend Rota’s result to a larger
class of spaces and operators. Apart from the particular type of un-
derlying space, the stringent condition in Rota’s theorem (supposing
that \(\mu(S) = 1\)) is \(r(T) = ||T||_1 = ||T||_\infty\), \(r(T)\) denoting the spectral radius
of \(T\) in \(L_1\) which is implicitly assumed to be one in [2]. From this,
we can drop the total finiteness of \(\mu\), the assumption \(||T||_1 = ||T||_\infty\) and
the requirement that \(L_\infty\) be invariant under \(T\) (\(T\) need indeed not
be defined on all of \(L_\infty\) when \(\mu(S)\) is infinite). More generally (Theo-
rem 1), the result is true for positive operators on any complex func-
tion space \(E\) of type \(L_p(S, \Sigma, \mu)\) or \(C(X)\) (\(X\) compact Hausdorff),
whenever \(T^* \psi \leq \psi\) for some strictly positive linear form \(\psi \in E'\). This
class includes all quasi-interior positive operators on \(C(X)\), for which
other spectral properties were obtained in [3]. More particularly, for
positive matrix operators on \(l_p\) satisfying the assumption above with
respect to some strictly positive linear form, the presence (assuming
\(r(T) = 1\)) of a single unimodular eigenvalue which is not a root of
unity, implies that the entire unit circle is in the point spectrum of \(T\)
(Theorem 2).

The assumption that \(T^* \psi \leq \psi\) for some strictly positive linear form,
in particular satisfied through \(||T||_1 = 1\) in Rota’s theorem, is by no
means necessary for the conclusion; it is made to ensure that

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\[\text{**E'** denotes the (topological) dual of E,} \, T^* \text{ the adjoint of T.}\]
\( \alpha f = T f \) \((|\alpha| = 1)\) implies \(|f| = T|f|\). To see how close this comes to what may be needed, we remark that every permutation matrix is the direct sum of a (possibly infinite) number of summands satisfying that assumption on a \( T \)-reducing subspace of \( l_p \).

2. In this section, we formulate the results and their immediate consequences. By \((S, \Sigma, \mu)\) we understand a measure space of finite or infinite measure, and by \( C(X) \) the \( B \)-space (under the sup-norm) of all continuous complex functions on a compact (Hausdorff) space \( X \). By a space of type \( C(X) \) we mean an ordered complex \( B \)-space which is (simultaneously algebraically, topologically, and order) isomorphic to some \( C(X) \) (but not necessarily isometric), and likewise for \( L_p(S, \Sigma, \mu) \) \((1 \leq p < \infty)\). \( J \) denotes the set of all rational integers.

**Theorem 1.** Let \( E \) be a space of type \( L_p(S, \Sigma, \mu) \) \((1 \leq p < \infty)\) or \( C(X) \), and let \( T \) be a positive linear operator on \( E \), with spectral radius \( 1 \), such that \( T'|p'|^p \) for some strictly positive linear form on \( E \). Then:

If \( \alpha f = T f, \ |\alpha| = 1, \) then \( \alpha^n|f|^n = T(|f|^n) \) for every \( n \in J \), where \( f = |f| g \).

**Corollary 1.** Under the conditions of the theorem, the point spectrum of \( T \) on the unit circle consists either of a finite number of (groups of) roots of unity, or it is dense.

It is not difficult to see that Theorem 1 contains Rota’s theorem as a special case. In fact, if \( ||T||_1 \leq 1 \), then \( T'\psi \leq \psi \) (and conversely), where \( \psi : f \to \|f\| d\mu \) is a strictly positive linear form on \( L_1(S, \Sigma, \mu) \).

There is another class of positive operators satisfying the condition of Theorem 1. A positive operator \( T \) on an ordered \( B \)-space \( E \) is called quasi-interior \([3]\) if there exists some \( \lambda_0 > r(T) \) such that \( TR(\lambda_0)x \) is a quasi-interior point of the positive cone \( K \) of \( E \), for each \( 0 \neq x \in K \); \( x_0 \in K \) is quasi-interior if the order interval \([0, x_0]\) is a total subset of \( E \). (For more details on quasi-interior maps, see \([3]\).)

**Corollary 2.** The assertion of Theorem 1 is valid for every quasi-interior positive map on a space of type \( C(X) \).

**Proof.** It is known (see, e.g., [3, Theorem 1, Corollary]) that for any positive operator on \( C(X) \) with spectral radius \( r \), there exists a nonzero linear form \( \psi \geq 0 \) satisfying \( r\psi = T'\psi \). Since for every \( f \in C(X) \), and \( \lambda > r \),

\[
\psi(f) \sum_{n=1}^{\infty} \frac{r^n}{\lambda^n} = \psi[TR(\lambda)f],
\]

\[\text{This is meant to include the case of empty point spectrum on } |\lambda| = 1.\]
it follows that $\psi$ is strictly positive when $T$ is quasi-interior and $r > 0$.

If $T$ is an operator on a space $l_p$ ($1 \leq p \leq \infty$), represented by a positive matrix, we obtain the following stronger result.

**Theorem 2.** Let $T$ be a positive matrix operator on some space $l_p$ ($1 \leq p \leq \infty$), satisfying the assumptions of Theorem 1. If $T$ has a unimodular eigenvalue which is not a root of unity, then the entire unit circle belongs to the point spectrum of $T$.

On the other hand, it results from the proof of Theorem 2 that if such a matrix has strictly positive diagonal entries, its point spectrum on the unit circle can at most contain the number 1.

3. The proof of Theorems 1 and 2 is divided into several steps. Unless any further distinction is needed, we denote by $E$ any Banach space of the type considered in Theorem 1.

(a) For any $f \in E$ and positive linear operator $T$ on $E$, $|Tf| \leq T|f|$. Let $s$ be a fixed element of $S$ (or $X$, respectively). We have $|Tf| (s) = (T\bar{f})(s)$ where $\bar{f} = fe^{i\delta}$ and $\delta = \delta(s)$ is suitably chosen. Let $\bar{f} = g + ih$, $g$ and $h$ denoting the real and imaginary parts of $f$, respectively. Now $T\bar{f} = Tg + iT\bar{h}$, and $T\bar{h}(s) = 0$ since $Tg$, $T\bar{h}$ are real-valued elements of $E$. Hence we have

$$T\bar{f}(s) = Tg(s) \leq T|f|(s)$$

since $|g| \leq |\bar{f}| = |f|$, whence it follows that $|Tf|(s) \leq T|f|(s)$. $s$ being arbitrary, we conclude that $|Tf| \leq T|f|$.

(b) If $\alpha f = Tf$ where $0 \neq f \in E$ and $|\alpha| = 1$, it follows that $|f| = |Tf|$ and hence, by (a), that $|f| \leq T|f|$. By assumption, there exists a strictly positive linear form satisfying $T^* \psi \leq \psi$; hence $\psi(|f|) \leq \psi(T|f|)$ and, therefore, $\psi(T|f| - |f|) = 0$. Since $\psi$ is strictly positive, $|f| = T|f|$.

(c) Let $H_0 = \{ t : |f(t)| > 0 \}$, and let $F$ denote the vector subspace of $E$ whose elements are of the form $|f|g$, $g \in G$, where $G$ is the vector space of all bounded $\Sigma$-measurable or all bounded continuous functions (accordingly as $E = L_p(S, \Sigma, \mu)$ or $E = C(X)$), on $H_0$. Since $|Tf|g| \leq ||g||_{\infty}T|f|$, $F$ is invariant under $T$. The formula

$$(Ug)(s) = |f(s)|^{-1} T|f|g(s) \quad (s \in H_0)$$

defines a positive endomorphism of $G$. Endowed with the sup-norm, $G$ is isomorphic (even isometric) with a space $C(Y)$, $Y$ compact. This is clear when $E = L_p$ (more precisely, one will consider the quotient space of $G$ modulo $\mu$-null functions), from the Gelfand-Naimark theorem; the same conclusion holds when $E = C(X)$, but here, more concretely, $G$ is isomorphic with $C(Y)$, where $Y$ is the Stone-Čech compactification of $X$. 

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compactification of $H_0 \subset X$. Thus in any case, we can associate with $f$, satisfying $|f| = T|f|$, a positive operator $U$ on $C(Y)$ such that $Ue = e$ (the constantly-one function on $Y$), and hence $\|U\| = 1$.

(d) We write $f = |f| g$, where $g$ is well defined on $H_0$; moreover, $|g| = e$ and $\alpha g = Ug$ since $\alpha f = Tf$ by assumption. We identify $G$ and $C(Y)$ in the spirit of the preceding paragraph. For each $s \in Y$, the mapping $h \mapsto (Uh)(s)$ defines a positive Radon measure $m_s$, of mass 1, on $Y$. The remainder of the proof rests on the following lemma.

**Lemma.** Let $g \in C(Y)$ satisfy $|g| = e$, and let $s \in Y$ be arbitrary. If $\alpha g = Ug$, $|\alpha| = 1$, then the support of $m_s$ is contained in $\{ t \in Y : g(t) = \alpha g(s) \}$.

The proof of this lemma is elementary, and will be omitted. Now let $n$ be any integer. Since for fixed $s$, $g(l) = \alpha g(s)$ on the support of $m_s$, it follows that

$$(Ug^n)(s) = \int g^n(l) \, dm_s(l) = \alpha^n g^n(s),$$

$s \in Y$, which proves that $\alpha^n g^n = Ug^n$. Translating back into $E$, we obtain $\alpha^n |f| g^n = T(|f| g^n)$, completing the proof of Theorem 1.

(e) It remains to prove Theorem 2. Let $A = (a_{ik})$ be the matrix representing the operator $T$, and let $\alpha x = Ax$ where $x \neq 0$ is a vector in $l_\infty$ and $|\alpha| = 1$, is not a root of unity. It follows from Theorem 1 that $|x| = A |x|$. As in (c) above, we associate with $A$ an operator $U$ on $l_\infty(H_0)$ where $H_0$ is the set of subscripts on which the coordinates of $x$ are nonzero; more precisely, $U$ is represented by the matrix $(u_{ik})$, where $u_{ik} = |x_i|^{-1} a_{ik} |x_k|$ for $i, k \in H_0$. Let $x_i = |x_i| v_i$ for $i \in H_0$, $v = (v_i)$. Further, denote by $\Gamma$ a set of representatives of the quotient group of the group of unimodular complex numbers over the subgroup $\{ \alpha^n : n \in J \}$. If $F_n = \{ j \in H_0 : v_j \in \alpha^n \Gamma \}$, $n \in J$, the $F_n$ are disjoint sets whose union is $H_0$.

From $\alpha v = Uv$ it follows, as in the lemma in (d), that if $v_i = \tau$ then $v_j = \alpha \tau$ for all $j$ such that $u_{ij} > 0$ ($i, j \in H_0$). (In fact, the lemma can be formally applied since $v \mapsto (Uv)_i$ is a Baire measure on the locally compact, discrete space $H_0$.) Denoting by $x^{(n)}$ the “characteristic” vector of $F_n$ in $l_\infty(H_0)$, we conclude that $Ux^{(n+1)} = x^{(n)}$. Moreover, the $x^{(n)}$ are mutually disjoint and $\sum \{ x^{(n)} : n \in J \} = e$, the constantly-one function on $H_0$.

Now let $\beta$ be any unimodular complex number. Letting

$$w = \sum_{n \in J} \beta^n x^{(n)},$$

4 At this point the assumption that $T$ be a matrix operator is essentially used.
it follows that $Uw = \beta w$, hence $\beta$ is in the point spectrum of $U$. Set
$y = (y_i)$, where $y_i = |x_i| w_i$ for $i \in H_0$ and $y_i = 0$ for $i \in H_0$. It is im-
mediate that $Ty = \beta y$ whence the theorem follows.

References

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