EMBEDDINGS OF PICARD VARIETIES

WILLIAM L. HOYT[1]

1. Let P be a projective space and $V \subset P$ a normal irreducible subvariety of dimension $\geqq 2$ defined over an algebraically closed field k. For a generic hyperplane H over k it is known that $V \cdot H$ is defined, irreducible, normal, and that the natural homomorphism ϕ of the Picard variety $P(V)$ of V into the Picard variety $P(V \cdot H)$ of $V \cdot H$ is purely inseparable and, in case dim $V \geqq 3$, also surjective ([5, Corollary, p. 169]; Professor Matsusaka has pointed out that the isomorphism asserted in this corollary may be purely inseparable). The problem of whether ϕ is birational remains open, but in this paper we prove the following:

THEOREM. *There exists $q(V)$ such that for almost all hypersurfaces H of degree $q \geqq q(V)$ the natural homomorphism of $P(V)$ into $P(V \cdot H)$ is birational.*

2. Our proof is based upon Matsusaka's construction of $P(V)$, the Lemma of Enriques-Severi-Zariski, and the lemmas below. We denote by L_m the space of forms (homogeneous polynomials) of degree m in $n+1$ indeterminants T_i, $0 \leqq i \leqq n$, $n = \dim P$, over the universal domain. If X is a positive homogeneous cycle in P, we denote by $\gamma(X)$ the Chow point of X; by $L_m(X)$ the subspace of L_m consisting of forms which vanish on supp X, the support of X; and by f_X the function defined on the integers such that $f_X(m) = 0$ for $m \leqq 0$ and $f_X(m) = \dim L_m - \dim L_m(X)$ for $m > 0$; then f_X is the Hilbert function for the radical ideal determined by supp X and hence coincides with a polynomial, which we denote by p_X, for large $m > 0$.

LEMMA 1. *Let W be a subvariety defined over k of a Chow bunch for r-cycles in P. Then there is a k-open set $W' \subset W$ such that $f_X = f_Y$ for all $\gamma(X), \gamma(Y) \in W'$.*

PROOF. For functions g from integers to integers let $\Delta^r g$ be defined by $\Delta^1 g(m) = g(m) - g(m-1)$ and $\Delta^{r+1} g = \Delta^1(\Delta^r g)$. One knows (cf. [6]) that (i) if $r = 0$, then $\Delta^1 p_X = 0$ and $\Delta^1 f_X$ is the Hilbert function for an irrelevant ideal, i.e., a primary ideal for $(T_0, T_1, \cdots, T_n)$; and (ii) if $r > 0$, then $\Delta^1 p_X = p_{X \cdot H}$ and $\Delta^1 f_X = f_{X \cdot H}$ for almost every hyperplane

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H in P. Hence one can show by induction on r that (iii) if $f_X(m) = p_X(m)$ for $r+2$ values $0<h\leqq m\leqq h+r+1$, then $\Delta^{r+1}f_X(h+r+1) = \Delta^{r+1}p_X(h+r+1)=0$ and hence $\Delta^{r+1}f_X(m)=0$ for all $m\geqq h+r+1$ (since $\Delta^{r+1}f_X(m)=0$ for $m>0$ if and only if $\Delta^{r+1}f_X$ corresponds to an ideal containing $(T_0, T_1, \cdots, T_n)^m$); and (iv) if $\Delta^{r+1}f_X(m)=0$ for all $m\geqq h$, then $f_X(m)=p_X(m)$ for all $m\geqq h$ (since $f_X(h+i)=f_X(h) + \sum_{1\leq j\leq i}\Delta^1 f_X(h+j)$ and $p_X(h+i)=p_X(h)+\sum_{1\leq j\leq i}\Delta^1 p_X(h+j)$, and by induction we may assume $\Delta^1 f_X(j)=\Delta^1 p_X(j)$ for $j\geqq h$). Finally (v) for fixed $h>0$ and generic $\gamma(X)\in W$ over k, one knows that the set of $\gamma(Y)\in W$ such that $f_Y(h)=f_X(h)$ is k-open in W (since $f_Y(h) = \dim A\cap\gamma(Y)\times L_h$, where A is the k-closed set consisting of pairs $(\gamma(Z), F)\in W\times L_h$ such that F vanishes on supp Z). Therefore, if h is chosen such that $f_X(j)=p_X(j)$ for $j\geqq h$ and if W' is the set of $\gamma(Y)\in W$ such that $f_X(j)=f_Y(j)$ for $0<j\leqq h+r+2$, then (i)-(v) imply that W' is k-open in W and that $f_X=f_Y$ for all $\gamma(Y)\in W'$.

3. LEMMA 2. *If W is as in Lemma 1 and if $r\geqq 1$, then there exists $q'(W)$ and for each $m\geqq q'(W)$ there exists a k-open set $A_m(W)\subset L_m$ such that $X\cdot H$ is defined for all $\gamma(X)\in W$ and all hypersurfaces H corresponding to forms $F\in A_m(W)$.*

PROOF. We use induction on dim W, the result being trivial when dim $W=0$. When dim $W>0$, we may assume that a generic $\gamma(X)\in W$ corresponds to an irreducible cycle X, since it is easily seen that if the lemma is true in this special case, then it is true in general. In this case we can find a k-open set $W''\subset W'$, where W' is as in Lemma 1, such that Y is irreducible for all $\gamma(Y)\in W''$. Let $B_m(W'')$ be the closure in L_m of $\bigcup_{\gamma(Y)\in W''} L_m(Y)$. Then $\dim B_m(W'')\leqq \dim L_m(X) + \dim W$; and $Y\cdot H$ is defined for all $\gamma(Y)\in W''$ whenever H corresponds to a form $F\in L_m - B_m(W'')$. However, since $\deg f_X=r\geqq 1$, there exists h such that $f_X(m)>\dim W$, $\dim B_m(W'')<\dim L_m$, and $L_m-B_m(W'')\neq\varnothing$ whenever $m\geqq h$. It follows that the lemma is true with $q'(W)=\max\{h, q'(C)\}$, where C runs over all the components of $W-W''$.

4. Let W be as in Lemma 2 with $r\geqq 1$; let $\gamma(X)\in W$ be generic over k, and let H be a hypersurface rational over k such that $X\cdot H$ is defined. Then we define $W_H=\mathrm{loc}_k\,\gamma(X\cdot H)$. If $X\cdot H$ has no multiple components, then $k(\gamma(X))\supset k(\gamma(X\cdot H))$ since these are the minimum fields of definition $\supset k$ for X and $X\cdot H$, resp.; and we denote by g_H the rational map from W to W_H determined by this inclusion.

LEMMA 3. *Let W be as in Lemma 2 with $r\geqq 1$ and such that a generic point corresponds to a cycle without multiple components. Then there* *exists* $q''(W)$ *and for each* $m \geqq q''(W)$ *there exists a k-open set* $C_m(W) \subset L_m$ *such that* g_H *is defined and birational for all hypersurfaces* H *corresponding to forms in* $C_m(W)$.

PROOF. Let $T^i=(T^i_j)$, $0 \leqq i \leqq r$, $0 \leqq j \leqq n$, denote $r+1$ sets of $n+1$ indeterminants; and let M (resp., N) denote the space of all multihomogeneous forms over the universal domain in the sets T^i, $0 \leqq i \leqq r$, (resp., in the sets T^i, $0 \leqq i \leqq r-1$) which are homogeneous of degree $=\deg Y$ for some $\gamma(Y) \in W$ in each set T^i. We identify each point $\gamma(Y) \in W$ (resp., $\gamma(Z) \in W_H$) with its Chow polynomial $F^Y(T^0, \cdots, T^{r-1}, T^r) \in M$ (resp., $F^Z(T^0, \cdots, T^{r-1}) \in N$) relative to some fixed choice of coordinates in P. We recall that if H is the hyperplane defined by $\sum u_j T_j = 0$ and if $Y \cdot H$ is defined, then $F^{Y \cdot H}(T^0, \cdots, T^{r-1}) = F^Y(T^0, \cdots, T^{r-1}, u)$ with $u=(u_j) \in P$. For each $v \in P$ let h_v be the rational map from M to N such that if $F \in M$, then $h_v(F)=F(T^0, \cdots, T^{r-1}, v) \in N$ whenever this latter form is nonzero. If $E_v \subset M \times N$ is the closure of the graph of h_v and if $F \in M$ is a point at which h_v is defined, then $E_v \cap M \times h_v(F) = (D_v \cup D_v(F)) \times h_v(F)$, where D_v is the set of $F' \in M$ such that $F'(T^0, \cdots, T^{r-1}, v) = 0$ and $D_v(F)$ is the set of forms $F'' \in M$ such that $F''(T^0, \cdots, T^{r-1}, v) = F(T^0, \cdots, T^{r-1}, v)$; both D_v and $D_v(F)$ are linear subspaces of M which are rational over any field of rationality for v and $h_v(F)$. For fixed $F_0 \in M$ it is clear that we can choose finitely many points $v_i \in P$, $1 \leqq i \leqq t$, rational over k, such that each h_{v_i} is defined at F_0 and $\bigcap_i (D_{v_i} \cup D_{v_i}(F_0)) = \{F_0\}$. If $F \in M$ is generic over $k(F_0)$, then also $\bigcap_i (D_{v_i} \cup D_{v_i}(F)) = \{F\}$; hence the product map $\prod h_{v_i} = h_{v_1} \times \cdots \times h_{v_t}$ from M to N^t is birational since F is rational over $k(\prod h_{v_i}(F))$; and $\prod h_{v_i}$ is biregular at F_0 since $\prod h_{v_i}(M - \bigcup D_{v_i})$ is nonsingular and $F \to F_0$ is the only specialization on M over $\prod h_{v_i}(F) \to \prod h_{v_i}(F_0)$.

If in the argument above we take $F_0 = \gamma(X)$, where $\gamma(X)$ is generic on W over k, then our definitions imply that the restriction of $\prod h_{v_i}$ to W is birational and coincides with the product map $\prod g_{H_i}$ from W to $\prod W_{H_i}$ where for $1 \leqq i \leqq t$, H_i is the hyperplane defined over k by the equation $\sum_j v_{ij} T_j = 0$ with $v_i = (v_{ij}) \in P$. Furthermore, we may choose the v_i so that each $X \cdot H_i$ has no multiple components and $X \cdot H_i \cdot H_j$ is defined for each $j \neq i$. This being the case, if $H = \sum H_i$, then $k(\gamma(X \cdot H)) = k(g_H(\gamma(X)))$ is the minimum field of definition $\supset k$ for $X \cdot H$ and hence is contained in $k(\gamma(X)) = k(\prod g_{H_i}(\gamma(X)))$. But if Z' is a conjugate over $k(g_H(\gamma(X)))$ of an irreducible component Z of some $X \cdot H_i$, then Z' is also a component of $X \cdot H_i$, since otherwise Z' must be a component of another $X \cdot H_j$ with $j \neq i$ and hence H_j contains both Z' and Z contrary to the hypothesis that $X\cdot H_i\cdot H_j$ is defined. Consequently, each $X\cdot H_i$ is rational over $k(g_H(\gamma(X)))$, and it follows that g_H is birational.

In case Lemma 2 holds with $q'(W)=1$, we can choose the H_i above so that $Y\cdot H_i$ is defined for each i and each $\gamma(Y)\in W$. Let H^* be a generic hypersurface of degree t over k, $G^*\subset W\times W_{H^*}$ the closure of the graph of g_{H^*}, $G\subset W\times W_H$ the closure of the graph of g_H with $H=\sum H_i$ as above, G' any specialization of G^* (in the appropriate ambient space) over $H^*\to H$, (z, w) any point in supp G', and $(\gamma(X), \gamma(X\cdot H^*))$ a generic point of G^* over everything in sight. Then there is a specialization $(\gamma(X), \gamma(X\cdot H^*))\to(z, w)$ over $H^*\to H$; and if $z=\gamma(Y)\in W$, it follows that $w=\gamma(Y\cdot H)\in W_H$ since $Y\cdot H$ is defined. Therefore, supp $G'\subset G$, and in fact $G'=G$ since $\mathrm{pr}_1\, G'=\mathrm{pr}_1\, G^*=W=\mathrm{pr}_1\, G$. Hence $\mathrm{pr}_2\, G^*=W_{H^*}$ and g_{H^*} is birational since $\mathrm{pr}_2\, G'=W_H$. An easy argument now shows that Lemma 3 holds with $q''(W)=t$ in this case.

Finally, if $q'(W)>1$, let $s\colon P\to P_1$ be the embedding of P in another projective space P_1 corresponding in the usual way to forms of degree $q'(W)$, and let $W_1=\mathrm{loc}_k\,\gamma(s(X))$, where $\gamma(X)$ is generic on W over k. Then $q'(W_1)=1$ since sections of a cycle Z in P by hypersurfaces of degree $q'(W)$ in P correspond to hyperplane sections of $s(Z)$ in P_1. Hence the lemma holds for W_1 by the argument above; and additional easy arguments now show (i) that the rational map g_H from W to W_H is birational for a generic hypersurface H of degree $q'(W)q''(W_1)$ (note that W and W_1 are birationally equivalent since X has no multiple components), and (ii) that the lemma holds for W with $q''(W)=(q''(W_1)+1)q'(W)$.

5. We now proceed to the proof of the theorem. Let V, P, and k be as in §1; let ϕ_V denote the canonical homomorphism of the group of divisors algebraically equivalent to zero on V onto $P(V)$; and for each positive divisor Y on V let $T_V(Y)$ denote the variety consisting of Chow points for all positive divisors in the complete linear system determined by Y on V. By [3, Proposition 9, p. 228], there exist a component U of a Chow bunch for positive divisors on V and a point $\gamma(X_0)\in U$ which is rational over k such that if $\gamma(X)$ is generic on U over k, then $P(V)$ is birationally equivalent to $\mathrm{loc}_k\,\gamma(T_V(X))$ under the map defined over k which sends $\phi_V(X-X_0)$ to $\gamma(T_V(X))$. For each normal irreducible subvariety $V'\subset V$ let $\phi_{V,V'}\colon P(V)\to P(V')$ denote the homomorphism such that $\phi_{V,V'}(\phi_V(Y))=\phi_{V'}(V'\cdot Y)$ whenever Y is a divisor algebraically equivalent to zero on V such that $V'\cdot Y$ is defined. (Since k is algebraically closed, one knows that

$k(\phi_V(Y))$ is the minimum field of rationality $\supset k$ for some $Y' \sim Y$ [4, Theorem 4, p. 59] and also that $k(\phi_V(Y))$ is contained in any field of rationality $\supset k$ for Y [3, Theorem 3, p. 231]; hence there is a natural inclusion $k(\phi_V(Y)) \supset k(\phi_{V'}(V' \cdot Y))$.) If $\dim V = r \geqq 3$, then since $\phi_{V',V''} \circ \phi_{V,V'} = \phi_{V,V''}$ whenever $V'' \subset V' \subset V$, our theorem will be proved if we can show that for almost all hypersurfaces H_1 of sufficiently high degree there exist hypersurfaces $H_2, H_3, \cdots, H_{r-1}$ such that the 1-cycle $C = V \cdot H_1 \cdot H_2 \cdot \cdots \cdot H_{r-1}$ is defined, irreducible, nonsingular, and $\phi_{V,C}$ is birational.

If $H_1, H_2, \cdots, H_{r-1}$ are independent generic hypersurfaces over k such that $\deg H_1$ is sufficiently large and $\deg H_{i+1}$ is sufficiently large compared with $\deg H_i$, if $K \supset k$ is a field of definition for $C = V \cdot H_1 \cdot H_2 \cdot \cdots \cdot H_{r-1}$, if $U_C = \operatorname{loc}_K \gamma\,(X \cdot C)$, where $\gamma(X)$ is generic in U over K, and if h_C is the rational map defined over K from U to U_C such that $h_C(\gamma(X)) = \gamma(X \cdot C)$, then Lemma 3 above and the Lemma of Enriques-Severi-Zariski [8, Theorem 4, p. 570] imply (i) that h_C is birational and (ii) that $h_C(T_V(X)) = T_C(X \cdot C)$. Let $C(d)$ denote the d-fold symmetric product of C; and for large d let $J_d(C) = \operatorname{loc}_K \gamma(T_C(\mathfrak{a}))$, where $\gamma(\mathfrak{a})$ is generic on $C(d)$ over K, denote the Chow model corresponding to d for the Jacobian variety of C [1, p. 468]. Then there exists a positive divisor $\mathfrak{a}_0$ on C which is rational over K such that $\gamma(T_C(X \cdot C + \mathfrak{a}_0)) \in J_d(C)$ for suitable d. Since h_C is biregular at $\gamma(X)$, $T_V(X)$ and $T_C(X \cdot C)$ have the same minimum field of definition $\supset K$. Hence $K(\gamma(T_V(X))) = K(\gamma(T_C(X \cdot C)))$. Also, if $\gamma(\mathfrak{a}_i) \in T_C(X \cdot C)$ and $\gamma(\mathfrak{b}_i) \in T_C(X \cdot C + \mathfrak{a}_0)$, $i = 1, 2$, are independent generic points over $K(\gamma(X))$, then it follows from [7, Chapter VIII, Theorem 10, p. 239], that

$$\begin{aligned} K(\gamma(T_C(X \cdot C))) &= K(\gamma(\mathfrak{a}_1)) \cap K(\gamma(\mathfrak{a}_2)) \supset K(\gamma(T_C(X \cdot C + \mathfrak{a}_0))) \\ &= K(\gamma(\mathfrak{b}_1)) \cap K(\gamma(\mathfrak{b}_2)) \supset K(\gamma(T_C(X \cdot C))). \end{aligned}$$

Therefore, $P(V)$ is birationally equivalent over K to a subvariety of $J_d(C)$ under the map which sends $\phi_V(X - X_0)$ to $\gamma(T_C(X \cdot C + \mathfrak{a}_0))$. Since this map is a composition of $\phi_{V,C}$ and a translation on $J_d(C)$, it follows that $\phi_{V,C}$ is birational. And since $J_d(C)$ and the graph and image of $\phi_{V,C}$ specialize to $J_d(C')$ and the graph and image of $\phi_{V,C'}$ over almost every specialization $C \to C'$ over k (cf. [2, Theorem 3, p. 188]), the theorem follows.

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BRANDEIS UNIVERSITY

A NOTE ON THEOREMS OF BURNSIDE AND BLICHFELDT

RICHARD BRAUER

1. The irreducible constituents of the tensor powers of a representation of a group. In his book [3], W. Burnside proved the following theorem (Theorem IV of Chapter XV):

THEOREM 1. *Let G be a finite group and let X be a faithful representation of G over the field* **C** *of complex numbers. Each irreducible representation X_λ of G appears as a constituent in some tensor power of X.*

Recently, R. Steinberg [5] has given a very simple proof of this theorem generalizing it at the same time. I shall give still another proof of Theorem 1. While this proof is less conceptual than Steinberg's proof, it is very short, and it refines the theorem in another direction.

THEOREM 1*. *Assume that the character χ of the representation X in Theorem 1 takes on a total of r distinct values $a_1, a_2, \cdots, a_r$ on G. Each irreducible character χ_λ of G appears as a constituent of one of the characters $\chi^0=1, \chi, \chi^2, \cdots, \chi^{r-1}$.*

PROOF. Let A_j be the set of elements $g\in G$ for which $\chi(g)=a_j$. Choose $g_j\in A_j$. If χ_λ is not contained in χ^i, then

$$|G|(\chi^i, \chi_\lambda) = \sum_j \chi^i(g_j) \sum_{g\in A_j} \bar{\chi}_\lambda(g) = 0.$$

If this holds for $i=0, 1, \cdots, r-1$, it follows from the nonvanishing

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