GENERALIZED LAURENT SERIES FOR SINGULAR SOLUTIONS OF ELLIPTIC PARTIAL DIFFERENTIAL EQUATIONS

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Introduction. Let $\mathcal{L}$ be a linear elliptic differential operator with analytic coefficients in a region $R$ of $\mathbb{R}^n$. Let $\mathcal{L}$ be the adjoint of $\mathcal{L}$. This paper extends the previous work of F. John on representation of a solution $u$ of $\mathcal{L}[u] = 0$, where $u$ has a singularity of finite order. A representation is developed here for a solution $v$ of $\mathcal{L}[v] = 0$, where $v$ has an isolated essential singularity. This representation is a generalization of the Laurent series. Here the summation over the $n$th powers is replaced by summation over the $n$th derivatives of a fundamental solution $K(x, z)$, of the operator $\mathcal{L}$. The representation in general is not unique.

Uniqueness of a suitably normalized representation is proved for the case in which $\mathcal{L}$ is homogeneous with constant coefficients. This gives rise to a theorem which for the three-dimensional Laplace operator reduces to the Maxwell-Sylvester theorem.

The general case. Let $v(x)$ be a solution of $\mathcal{L}[v(x)] = 0$, which has an isolated essential singularity at $x = y$, an interior point of the real region $R$, but is otherwise regular in the deleted region $R$. Let $\mathcal{L}$ be of order $m$.

Let $\mathcal{D} \subset R$ be an open annular domain about the point $x = y$. Let $S_1$ be the sphere bounding the outer ball $B_1$ of $\mathcal{D}$ and let $S_2$ be the sphere bounding the inner ball $B_2$ of $\mathcal{D}$. Both $B_1$ and $B_2$ have the point $x = y$ as center and $B_1 \supset B_2$. We further assume that $B_1$ is so small that $K(x, z)$, a fundamental solution of $\mathcal{L}$, is analytic for $x \neq z$ in $B_1$.

**Theorem I.** For $z \in \mathcal{D}$, $v(z)$ permits the following representation:

$$v(z) = \omega(z) - \sum_{r=0}^{\infty} \sum_{|i| = r} A_i D^i K(x, z) |_{x=y},$$

where $\omega(z)$ is analytic for $z \in B_1$, $A_i$ are constants depending on $S_2$, $|z| = i_1 + i_2 + \cdots + i_n$ and

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1 I am indebted to Professor Fritz John for suggesting this problem.

2 See [1].

3 See [2, pp. 514–521].
\[ D^i = \frac{\partial^{|\iota|}}{\partial x_1^{i_1} \partial x_2^{i_2} \cdots \partial x_n^{i_n}}. \]

**Proof.** We know that there exists a ball \( B_1 \subseteq B_2 \) with center at \( x = y \) such that \( K(x, z) \) has a Taylor series expansion with respect to \( x \), about \( x = y \), for \( x \in B_3 \), \( z \in \mathcal{D} \). Let \( S_3 \) be the sphere bounding \( B_3 \) and let \( \bar{R} \) be the annular region defined by \( S_1 \) and \( S_3 \). \( \bar{R} \) has the boundary \( \beta = S_1 \cup S_3 \). Applying Green’s identity to the operator \( L \) over \( \bar{R} \) we get for \( z \in \mathcal{D} \):

\[
v(z) = \int_\beta M[K(x, z), v(x)]dS_x
\]

(1)

\[
= \int_{x \in S_1} M[K(x, z), v(x)]dS_x - \int_{x \in S_3} M[K(x, z), v(x)]dS_x,
\]

where \( M \) is a bilinear operator, and \( dS_x \) denotes integration over the surface of the boundary.

\[
\omega(z) = \int_{x \in S_1} M[K(x, z), v(x)]dS_x
\]

is clearly analytic for \( z \in \mathcal{D} \), since \( x \in S_1 \) and therefore \( x \neq z \).

For \( z \in \mathcal{D} \), \( x \in S_1 \), \( K(x, z) \) has the Taylor series expansion about \( x = y \):

\[
K(x, z) = \sum_{|\iota| = 0}^{\infty} \sum_{|\iota| = 0}^{\infty} \frac{1}{i!} (x - y)^i D_y^\iota K(y, z),
\]

where \( D_y^\iota K(y, z) = D^\iota K(x, z) \big|_{x \to y} \); \( (x - y)^i = (x_1 - y_1)^{i_1} \cdots (x_n - y_n)^{i_n} \); and \( i! = (i_1)! \cdots (i_n)! \). Equation (1) then becomes:

\[
v(z) = \omega(z) - \int_{x \in S_1} M \left[ \sum_{|\iota| = 0}^{\infty} \sum_{|\iota| = 0}^{\infty} \frac{1}{i!} (x - y)^i D_y^\iota K(y, z), v(x) \right] dS_x.
\]

Since \( M \) is a linear operator and the series is uniformly convergent:

\[
v(z) = \omega(z) - \sum_{|\iota| = 0}^{\infty} \frac{1}{i!} \int_{x \in S_1} D_y^\iota K(y, z) \int_{x \in S_1} M[(x - y)^i, v(x)] dS_x.
\]

Let

\[
A_i = \frac{1}{i!} \int_{x \in S_1} M[(x - y)^i, v(x)] dS_x.
\]

\( A_i \) is a constant. We then have as a final expression:
(2) \[ v(z) = \omega(z) - \sum_{r=0}^{\infty} \sum_{|s|=-r} A_s D^s_y K(y, z). \]

It would be desirable to normalize equation (2), i.e., to obtain a form of equation (2) in which the \( A_s \) are independent of the spheres \( S_2 \) and \( S_3 \). Equation (2) would then be unique and hold for any \( z \neq y, z \in B_1 \); because for any such \( z \), we can find a \( B_2 \) and a \( B_3 \) with radius so small that the required Taylor series expansion for \( K(x, z) \) will exist.

We shall succeed in giving such a normalization of equation (2) only for the special case of \( \mathcal{L} \) homogeneous with constant coefficients, i.e.,

(3) \[ \mathcal{L} = \sum_{|s|=m} b_s D^s, \]

where the \( b_s \) are real constants. Let the coefficient of \( \frac{\partial^m}{\partial x_1^m} \) in \( \mathcal{L} \) be \( b_1 \). Since \( \mathcal{L} \) is elliptic \( b_1 \) is not zero. We know that there exists a fundamental solution of \( \mathcal{L} \) of the form \( K(y - x) \),\(^4\) we shall use this fundamental solution in Theorem II.

Theorem II.\(^5\) Let \( v(x) \) be a solution of \( \mathcal{L}[v(x)] = \sum_{|s|=m} b_s D^s v(x) = 0 \) for \( x \in B_1, \; x \neq y \). Let \( v \) have an isolated essential singularity at \( x = y \). Then there exists a representation of \( v(z) \) of the form:

\[ v(z) = \omega(z) - \sum_{r=0}^{\infty} \sum_{|s|=-r} A_s D^s_y K(y - z), \quad \text{for } z \neq y, \; z \in B_1, \]

where \( A_s = 0 \), whenever \( D^s_y \) contains \( \frac{\partial^m}{\partial y_1^m} \) as factor (i.e., whenever \( s_1 \geq m \)); and this representation is unique.

Proof. For \( z \neq y, \; z \in \Omega \), using equation (2) we can write:

(4) \[ v(z) = \omega_1(z) - \sum_{r=0}^{\infty} \sum_{|s|=-r} A_s D^s_y [K(y - z)]. \]

We will prove that each operator term \( \sum_{|s|=m} A_s D^s_y \) in the above series, with \( \nu \geq m \) can be written in the following form when it is applied to any solution \( u \) of \( \mathcal{L}[u] = 0 \).

\(^4\) See [1, pp. 298–303].

\(^5\) This is a generalization of the Maxwell-Sylvester theorem for the case of the three-dimensional Laplacian. See [2, pp. 514–521].
where \( J_s \) are constants. We will also prove that equation (5) is unique.

From equation (3) we have:

\[
D^j u = \frac{1}{b_j} \left[ \mathcal{L} - \sum_{|i|=m} b_i D^i \right] u = -\sum_{|i|=m} \frac{b_i D^i u}{b_I}.
\]

The operator term \( \sum_{|k|\to p} A_k D^k \) can be written as a polynomial in powers of \( \partial/\partial x_1 \), i.e.,

\[
\sum_{|k|\to p} A_k D^k = \sum_{j=0; i=j+|\mu|=\sigma+\rho}^p G_{n+j} \frac{\partial^i}{\partial x_1^i},
\]

where \( G_{n+j} \) are constants, \( \rho \) a non-negative integer which is the highest order of \( \partial/\partial x_1 \) in \( \sum_{|k|\to p} A_k D^k \). For solutions \( u \) of \( \mathcal{L}[u] = 0 \) and \( p \geq m \), using equation (6), we can reduce the \( \rho \)th order polynomial operator in \( \partial/\partial x_1 \) in equation (7) to a \((p-1)\)th order polynomial in \( \partial/\partial x_1 \). It is then seen that after \( p - m + 1 \) steps equation (7), when applied to solutions \( u \) of \( \mathcal{L}[u] = 0 \) reduces to:

\[
\sum_{|k|\to p} A_k D^k u = \sum_{j=0; i=j+|\mu|=\sigma+\rho}^p H_{n+j} \frac{\partial^i}{\partial x_1^i} u = \sum_{|\alpha|=\sigma; x_1^{\alpha}} J_s D^x u,
\]

where \( H_{n+j}, J_s \) are constants.

We now prove uniqueness. Suppose we had the two identities, for all \( u \) with \( \mathcal{L}[u] = 0 \):

\[
\sum_{|k|\to p} A_k D^k u = \sum_{|\alpha|=\sigma; x_1^{\alpha}} J_s D^x u
\]

and

\[
\sum_{|k|\to p} A_k D^k u = \sum_{|\alpha|=\sigma; x_1^{\alpha}} J_s' D^x u;
\]

then

\[
\mathcal{L}[u] = \sum_{|\alpha|=\sigma; x_1^{\alpha}} (J_s - J_s') D^x u = 0.
\]

Equation (10) holds for all solutions \( u \) of \( \mathcal{L}[u] = 0 \). Consider solutions \( u \) of the form:

\[
u(x) = f(x_1)e^{\sigma_1 x_1 + \cdots + \sigma_n x_n},
\]

where the \( \sigma_i \) are constants. For a given set of \( \sigma_i \), \( \mathcal{L}[u(x)] = 0 \) is an \( m \)th order differential equation in the variable \( x_1 \), and there are \( m \)

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\(^6\) The index \( \mu + j = (\mu + j, \mu_n, \cdots, \mu_0) \).
independent solutions, \( f(x_1) \). If we apply \( \mathcal{L} \) to \( u(x) \) as defined in equation (11), then for a given set of \( \sigma \), we get either at most \( m-1 \) independent solutions for \( f(x_1) \) or that the coefficients of all derivatives of \( f(x_1) \) in \( \mathcal{L} \) vanish.

Since equation (10) is to hold for all solutions \( u \) of \( \mathcal{L}[u] = 0 \), it is impossible for \( \mathcal{L}_v[K(y-z)] = 0 \) to give only \( m-1 \) solutions for \( f(x_1) \). On the other hand, if the coefficients of all orders of \( \partial / \partial x_1 \) vanish in equation (10) for all sets of \( \sigma \), they vanish identically. We therefore conclude that \( J = J' \), i.e., equation (5) is unique.

We now apply identity (5) to \( u = K(y-z) \), which satisfies \( \mathcal{L}_v[K(y-z)] = 0 \) for \( s \neq y \), since \( L \) is self-adjoint. Substituting in equation (4) we get the normalized representation:

\[
\psi(z) = \omega_1(z) - \sum_{r=0}^{\infty} \sum_{|s|=m} J_r D_r^k K(y-z) \quad \text{for } z \neq y, z \in \mathcal{D},
\]

where we have defined \( A_r = J_r \) for \( |s| < m \).

We now prove that equation (12) is a unique representation of \( \psi(z) \).\(^7\) In particular it is independent of the construction leading to the coefficients \( A_r \). For suppose we had another representation of \( \psi(z) \):

\[
\psi(z) = \omega_2(z) - \sum_{r=0}^{\infty} \sum_{|s|=m} J'_r D_r^k K(y-z), \quad \text{for } z \neq y, z \in \mathcal{D}',
\]

where \( \mathcal{D}' \) is some annular region about the point \( x = y \) such that \( \mathcal{D} \cap \mathcal{D}' = \mathcal{D}'' \) is a nonempty set. Let the outer ball of \( \mathcal{D}' \) be \( B''_0 \) with radius \( r_0 \), and the inner ball be \( B''_1 \) with radius \( r_1 \).

Subtracting equation (13) from equation (12) we get for \( z \neq y, z \in \mathcal{D}'' \):

\[
\phi(z) = \omega_1(z) - \omega_2(z) = \sum_{r=0}^{\infty} \sum_{|s|=m} (J_r - J'_r) D_r^k K(y-z).
\]

We want to prove that \( J = J' \) and \( \omega_1(z) = \omega_2(z) \). We know the form of the fundamental solution \( K(y-z) \).\(^8\)

\[
K(y-z) = \begin{cases} 
\rho^{m-n} A \left( \frac{z-y}{\rho} \right) & \text{for odd } n, \\
\rho^{m-n} \left[ B \left( \frac{z-y}{\rho} \right) \log \rho + C \left( \frac{z-y}{\rho} \right) \right] & \text{for even } n,
\end{cases}
\]

\(^7\) Unique within the choice of \( x_1 \).

\(^8\) See [1, pp. 298–303].
where \( \rho = |z-y| \), and \( A, B, C, K(y-z) \) are analytic for all real \( y, z \) with \( y \neq z \). For \( n \) even \( \rho^{m-n}B((z-y)/\rho) \) is a polynomial in \( z-y \). Then \( D^*_nK(y-z) \) has the form:

\[
D^*_nK(y-z) = \begin{cases} 
\rho^{m-n-1}E_n\left(\frac{z-y}{\rho}\right) & \text{for odd } n, \\
\rho^{m-n-1} \left[ H_n\left(\frac{z-y}{\rho}\right) \log \rho + G_n\left(\frac{z-y}{\rho}\right) \right] & \text{for even } n,
\end{cases}
\]

\( E_n, H_n, G_n \) are regular for all real \( z \neq y \). We can then write equation (14) in the form

\[
\phi(z) = \sum_{\gamma=0}^{\infty} \sum_{|z\pm\gamma|<\infty} (J_\gamma - J'_\gamma) D^*_nK(y-z)
\]

\[
= \sum_{\gamma=0}^{\infty} \rho^{m-n}N_\gamma\left(\frac{z-y}{\rho}\right) - Q\left(\rho, \frac{z-y}{\rho}\right) \log \rho.
\]

Let

\[
\eta = \frac{z-y}{\rho};
\]

then

\[
\phi(y + \rho \eta) = \sum_{\gamma=0}^{\infty} \rho^{m-n}N_\gamma(\eta) - Q(\rho, \eta) \log \rho,
\]

where \( Q(\rho, \eta) \) is a polynomial in \( z-y \) and \( Q=0 \) for odd \( n \).

Consider an analytic continuation of equation (16) from the real \( \rho \) to the complex \( \zeta = \rho + i\xi \). For any fixed \( \eta \), both \( \phi(y+i\xi \eta) \) and \( \sum_{\gamma=0}^{\infty} \rho^{m-n}N_\gamma(\eta) \) are univalued functions, analytic in \( \xi \), and \( Q(\xi, \eta) \log \xi \) is a multivalued function in \( \mathbb{D}' \). Since this is true for every fixed \( \eta \), we conclude that for both odd and even \( n \),

\[
\phi(y + \rho \eta) = \sum_{\gamma=0}^{\infty} \rho^{m-n}N_\gamma(\eta).
\]

For any fixed \( \eta \), the right-hand side of equation (17) converges not only for \( \tau_0 > \rho > \tau_1 \) but for all \( \infty > \rho \geq \tau_1 \). This is so because the series part of equation (17) becomes a power series in negative powers of \( \rho \), except possibly for a finite number of positive powers in the case of \( m-n \). \( \phi(y+\rho \eta) \) is then analytic for all \( \infty > \rho \geq 0 \). From the analyticity of \( \phi(y+\rho \eta) \) in \( \infty > \rho \geq 0 \), we conclude, that the portion of the series in equation (17) containing negative powers of \( \rho \) must vanish, i.e.,
(18) \[ \sum_{r=0; m-n-r<0}^{\infty} \rho^{m-n-r}N_r(\eta) = 0. \]

For \( m-n<0 \) this means that \( \phi(z) = 0 \). When \( m-n \geq 0 \) we get in addition:

(19) \[ \sum_{r=0; m-n-r \geq 0}^{m-n} \rho^{m-n-r}N_r(\eta) = \phi(y + \rho \eta). \]

We will take these two cases separately. First for equation (19), using equation (15), we get:

(20) \[ \phi(z) = \sum_{r=0; m-n-r \geq 0}^{m-n} \sum_{|s|=s} (J_s - J'_s) D^s y K(y - z) = L_y[K(y - z)] \]

Since \( |s| \leq m-n \), from the form of \( D^s y K(y - z) \) we see that equation (20) holds even at \( y = z \). Let \( B \) be a ball containing \( y \) as an interior point. Let \( \psi(\eta) \) be any regular function which vanishes in a neighborhood of \( \beta \), the boundary of \( B \). By Green's identity

\[ \psi(\eta) = \int_B L[\psi(z)]K(\eta - z) ds; \]

then

(21) \[ L[\psi(\eta)]|_{s=z} = L_y[\psi(y)] = \int_B L[\psi(z)]\phi(z) ds = \int_B \psi(z)L[\phi(z)] dz. \]

Since \( L \) is self-adjoint and \( \phi \) is a solution of \( L[\phi] = 0 \), equation (21) implies that every regular function \( \psi \), which vanishes in a neighborhood of \( \beta \) has the property that at \( x = y \), \( L[\psi] = 0 \). We then conclude that \( L[\psi] = 0 \) and therefore from equation (20) \( \phi \equiv 0 \), i.e., \( \omega_1(x) = \omega_2(x) \).

Also for \( m-n \geq 0 \), and \( m-n \geq |s| \geq 0 \), we have \( J_s = J'_s \).

We now return to equation (18). Since for every fixed \( \eta \) equation (18) is a power series its terms vanish separately. From equation (15) this means that:

(22) \[ \sum_{|s|=s; m-n-s<0} (J_s - J'_s) D^s y K(y - z) = 0. \]

This could be written:

(23) \[ \sum_{|s|=s; m-n-s<0} (J_s - J'_s) D^s y K(y - z) = \pm \sum_{|s|=s; m-n-s<0} (J_s - J'_s) D^s y K(y - z) = P_s[K(y - z)] = 0. \]
$P_z$ is a linear homogeneous operator with constant coefficients, and equation (23) holds for all $z$ such that $r_z \leq |y - z|$. But $K(y - z)$ is analytic for all real $z$ with $z \neq y$. Then equation (21) holds for all real $z$ with $z \neq y$.

Let $G$ be a simply connected open finite region. Let $\beta_\partial$ be the boundary of $G$. Then a regular solution $f$, of $\bar{L}[f] = 0$ has a representation in $G$:

$$f(z) = \int_{\beta_\partial} M[K(\xi - z), f(\xi)]dS_\xi$$

and so

$$P_z[f(z)] = \int_{\beta_\partial} M[P_z[K(\xi - z)], f(\xi)]dS_\xi = 0 \text{ in } G.$$ 

Then every regular solution $f$ of $\bar{L}[f] = 0$ in $G$ is also a solution of $P_z[f(z)] = 0$ in $G$. But $P_z$ has the property that all $D_z^m$ in $P_z$ have $s_1 \leq m - 1$. It has been proved previously that not every $f(z)$ which solves $\bar{L}_s[f(z)] = 0$ can solve $P_z[f(z)] = 0$ unless $P_z \equiv 0$. We then conclude that $J_s = J'_s$ for all $|s| \geq 0$, and that equation (12) is a unique representation of equation (2). Furthermore, this representation holds right up to the singularity, since in our original construction we may now take $r_2 > 0$, the radius of $S_2$ as small as we wish.

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