A NOTE ON GROUPS OF PRIME POWER EXPONENT SATISFYING AN ENGEL CONGRUENCE

GERALD LOSEY

1. In [3] Kostrikin has proved that a Lie algebra of characteristic \( p \), \( p \) prime, satisfying the \( n \)th Engel condition, \( n \leq p \), is locally nilpotent. Using this result he has shown that in a group of exponent \( p \) generated by \( r \) elements, \( r \) finite, one has \( G_m = G_{m+1} \) (where \( G_i \) denotes the \( i \)th term of the lower central series of \( G \)) and \( m \) is dependent only on \( p \) and \( r \). In this note we shall apply Kostrikin's result to groups of exponent \( p^n \).

2. Let us denote by \( (x, y) \) the group commutator \( x^{-1}y^{-1}xy \). We define the symbol \((x, y; n)\) recursively by

\[
(x, y; 1) = (x, y), \quad (x, y; n + 1) = ((x, y; n), y),
\]

that is, \((x, y; n)\) is a left normed commutator of weight \( n + 1 \). A group \( G \) is said to satisfy the \( n \)th Engel congruence if \( (x, y; n) \equiv 1 \) mod \( G_{n+2} \) for all \( x, y \in G \). In a group of exponent \( p \) the \((p-1)\)st Engel congruence holds (P. Hall, [1]).

3. If \( G = H_1 \supseteq H_2 \supseteq H_3 \supseteq \cdots \) is a descending central series for \( G \) having the property that \( (H_i, H_j) \subseteq H_{i+j} \) for all \( i \) and \( j \) then one can construct, in the canonical fashion, the Lie ring

\[
L(H_i) = \sum (H_i/H_{i+1}), \quad \text{direct sum}
\]

(cf. Lazard, [4], and Higman, [2]). In particular, if \( \{G_i\} \) is the lower central series of \( G \) we obtain the Lie ring \( L(G_i) \) used by Higman, [2], and Kostrikin, [3]. It has been shown by Higman, [2], that if \( G \) satisfies the \( n \)th Engel congruence then the relation \( n!xy^n = 0 \) is satisfied for all \( x, y \in L(G_i) \).

4. Since \( G_i \subseteq H_i \) for all \( i \), we have \( G_i H_{i+1}/H_{i+1} \) a subgroup of \( H_i/H_{i+1} \). Hence

\[
L^*(H_i) = \sum (G_i H_{i+1}/H_{i+1}), \quad \text{direct sum},
\]

is a subgroup of \( L(H_i) \). For \( x \in G_i \) and \( y \in G_j \) we have \((x, y) \in G_{i+j} \subseteq G_{i+j} H_{i+j+1} \) and so it follows that \( L^*(H_i) \) is a Lie subring of \( L(H_i) \). Moreover, the mappings \( \phi_i: G_i/G_{i+1} \rightarrow G_i H_{i+1}/H_{i+1} \) given by \( xG_{i+1} \rightarrow xH_{i+1} \) are homomorphisms. The mapping \( \phi = \sum \phi_i \) is then easily

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seen to be a homogeneous Lie ring homomorphism of $L(G_i)$ onto $L^*(H_i)$. Thus $L^*(H_i)$ is a subring of $L(H_i)$ and a homomorphic image of $L(G_i)$.

5. We can now state and prove our result.

**Theorem.** Let $G$ be a finitely generated group of exponent $p^a$, $p$ prime, such that

(i) $G$ satisfies the $(p-1)^{st}$ Engel congruence,

(ii) for some integer $m>0$, $G_m$ has exponent $p$.

Then there exists an integer $n$ such that $G_n = G_{n+1}$.

**Proof.** In [5] Zassenhaus shows that the sequence of subgroups $H_k = \prod_{i,j} G_{ij}^p$, the product taken over all $i$ and $j$ such that $ip^j \geq k$, form a descending central series for $G$ satisfying the conditions $(H_r, H_s) \subseteq H_{r+s}$ and $H_r^p \subseteq H_{r+p} \subseteq H_{r+1}$. Hence $L(H_i)$ is a Lie algebra of characteristic $p$ and, hence, so is $L^*(H_i)$. Since the $(p-1)^{st}$ Engel congruence holds in $G$, the relation $(p-1)! xy^{p-1} = 0$ holds in $L(G_i)$ and hence also in the homomorphic image $L^*(H_i)$. Since $L^*(H_i)$ has characteristic $p$, this implies that the $(p-1)^{st}$ Engel condition, $xy^{p-1} = 0$, holds in $L^*(H_i)$. Since $G$ is finitely generated, $L^*(H_i)$ is also finitely generated and so, by Kostrikin’s result, $L^*(H_i)$ is nilpotent. Thus for some integer $N$ we have $G_n \subseteq H_{n+1}$ or

$$G_n \subseteq G_{n+1} G_{n}^{p} G_{n-1} \cdots = \prod G_i^{p^j}$$

for all $n \geq N$. By taking $n$ large enough we can ensure that in (1) each term $G_k$ with $k < m$ occurs with exponent $\geq p^a$. Then (1) will reduce to $G_n \subseteq G_{n+1}$, that is, $G_n = G_{n+1}$.

6. This result is not as strong as one might hope for. We would like to have the index $n$ depend only on $p^a$, $m$ and the number of generators of $G$. However it is not possible to assert the existence of a universal group $G$ generated by $r$ elements and satisfying the conditions of the theorem and having any other such group as a homomorphic image. Since the sequence $\{H_i\}$ of Zassenhaus is the most rapidly descending central series such that $H_i/H_{i+1}$ has exponent $p$, it seems unlikely that a stronger result can be obtained using the method of this note unless one can strengthen Kostrikin’s theorem.

**References**

A GENERALIZATION OF THE CARTAN-BRAUER-HUA THEOREM

CHARLES J. STUTH

Let $K$ be a division ring. Then $K'$ will denote the multiplicative group of $K$. If $S$ is a subset of $K$, then $S'$ will denote the division ring generated by $S$, and $C(S)$ will denote the centralizer of $S$ in $K$. If $G$ is a subgroup or a subdivision ring contained in $K$, then $Z(G)$ is the center of $G$. If $x$ and $y$ are elements of $K'$, then $(x, y) = yx^{-1}y^{-1}$ and $x^y = yxy^{-1}$. If $b$ is an element of a group $A$, then $Cl(A, b)$ will denote the group which is generated by the conjugate class of $b$ in $A$.

A set is central in $K$ if each of its elements is in $Z(K)$. A group $G$ is $n$-subnormal in a group $H$ if there are groups $G_1, \ldots, G_{n-1}$ such that $G = G_n \triangleright G_{n-1} \triangleright \cdots \triangleright G_1 \triangleright G_0 = H$. A group $G$ is subnormal in a group $H$ if $G$ is an $n$-subnormal subgroup of $H$ for some $n$. A group $G$ is invariant under a group $H$ if $gh \in G$ for all $g \in G$, all $h \in H$. A non-central subgroup $G$ is of type I if $G$ is invariant under a noncentral subgroup of $K'$. $K$ has the property $P_n$ if for every subdivision ring $H$ of $K$ such that $H'$ is invariant under a noncentral $n$-subnormal subgroup of $K'$ it follows that $H$ is central or $H = K$.

The Cartan-Brauer-Hua Theorem [2] states that a division ring has property $P_0$. Herstein and Scott [1] generalized this to $P_1$. Schenkman and Scott [5] extended the Cartan-Brauer-Hua Theorem by showing that a division ring has property $P_n$ for all $n$ if each of its subdivision rings which is invariant under a noncentral subgroup is normal in some subnormal subgroup of the division ring.

Theorem 1 of this paper shows that a division ring has property $P_n$ for all $n$. Then results are developed from this concerning the subnormal subgroups of $K'$ and more generally for the subgroups of type I in $K'$.