A CLASS OF UNIVALENT FUNCTIONS

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1. Introduction. The condition Re \( f'(z) > 0 \) is known to be sufficient for the univalence of an analytic function in any convex domain. In a recent paper \([3]\) the author investigated the class of functions which satisfy Re \( f'(z) > 0 \) for \( |z| < 1 \) and are normalized by \( f(0) = 0, f'(0) = 1 \). In this paper we study the subclass denoted by \( F \) and defined by the condition \( |f'(z) - 1| < 1 \) for \( |z| < 1 \). Some of our results have already been proven for the particular functions in \( F \) whose coefficients satisfy \( \sum_{n=2}^{\infty} n|a_n| \leq 1 \). Particular reference should be made of a paper by Schild \([5]\).

2. Distortion theorems. Suppose that \( f(z) \in F \). By applying Schwarz's lemma to the function \( f'(z) - 1 \) we obtain \( |f'(z) - 1| \leq |z| \). This gives the estimates \( 1 - |z| \leq |f'(z)| \leq 1 + |z| \). Bounds for \( |f(z)| \) can be obtained from \( |f'(z) - 1| \leq |z| \) by integration, as follows. Integrating along the line segment from 0 to \( z \) we may write

\[
f(z) - z = \int_0^z (f'(s) - 1) \, ds = z \int_0^1 (f'(tz) - 1) \, dt.
\]

\[
|f(z) - z| \leq |z| \int_0^1 |f'(tz) - 1| \, dt \leq |z| \int_0^1 t |z| \, dt = (1/2) |z|^2.
\]

From this estimate for \( |f(z) - z| \) we immediately obtain

\[
|z| - (1/2) |z|^2 \leq |f(z)| \leq |z| + (1/2) |z|^2.
\]

Each estimate is precise only for the functions \( f(z) = z + a_2 z^2 \), where \( |a_2| = 1/2 \).

The bounds for \( |f(z)| \) imply the following theorem.

**Theorem 1.** Each function in \( F \) assumes every complex number in the circle \( |w| < 1/2 \). No values outside of the circle \( |w| < 3/2 \) are assumed.

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1 Most of the results in this paper are contained in the author's dissertation, which was completed at the University of Pennsylvania in June, 1961.

2 The preparation of this paper was partially supported by the National Science Foundation under the grant NSF-GP-161.
3. **An area theorem.** If \( g(z) = \sum_{n=0}^{\infty} b_n z^n \) is regular for \( |z| < 1 \) and \( |g(z)| \leq 1 \) then \( \sum_{n=0}^{\infty} |b_n|^2 \leq 1 \) [1 p. 7]. Applying this estimate to \( f'(z) - 1 \) where \( f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in F \) we get \( \sum_{n=2}^{\infty} n^2 |a_n|^2 \leq 1 \). We will use this estimate in the next theorem. It also shows that the coefficients of functions in \( F \) satisfy \( |a_n| \leq 1/n \) for \( n = 2, 3, \ldots \) and equality for a given \( n \) holds only for functions of the form \( f(z) = z + a_n z^n \).

**Theorem 2.** The area of the image of \( |z| < 1 \) under each function in \( F \) satisfies \( A \leq (3/2)\pi \).

**Proof.** Suppose that \( f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in F \) and \( 0 < r < 1 \). Let \( D_r \) and \( D \) be the images of \( |z| < r \) and \( |z| < 1 \) under \( f(z) \) and let \( A_r \) and \( A \) be the areas of \( D_r \) and \( D \). The area of \( D \) exists since \( D \) is open and bounded. \( A_r \) is given by the following well-known formula.

\[
A_r = \pi \left( r^2 + \sum_{n=2}^{\infty} n |a_n|^2 r^{2n} \right).
\]

We will prove that

\[
A = \pi \left( 1 + \sum_{n=2}^{\infty} n |a_n|^2 \right).
\]

The convergence of this series follows from the fact that \( \sum_{n=2}^{\infty} n^2 |a_n|^2 \) converges. Let \( \{r_n\} \) be any increasing sequence of positive numbers such that \( r_n \rightarrow 1 \). Then \( \{D_{r_n}\} \) is an increasing sequence of sets whose union is \( D \). From the theorem in measure theory \( A_{r_n} \rightarrow A \). However the convergence of (1) implies that \( A_r \) is continuous for \( 0 \leq r \leq 1 \). Consequently \( A_{r_n} \rightarrow A_1 \). This proves (1).

From (1) and \( \sum_{n=2}^{\infty} n^2 |a_n|^2 \leq 1 \), we obtain

\[
A = \pi \left( 1 + \sum_{n=2}^{\infty} n |a_n|^2 \right) \leq \pi \left( 1 + (1/2) \sum_{n=2}^{\infty} n^2 |a_n|^2 \right) = (3/2)\pi.
\]

This proves \( A \leq (3/2)\pi \) and \( A = (3/2)\pi \) only for the functions \( f(z) = z + a_2 z^2 \) where \( |a_2| = 1/2 \).

4. **The boundary of the image domain.**

**Theorem 3.** Each function in \( F \) maps \( |z| < 1 \) onto a domain whose boundary is a rectifiable Jordan curve.
Proof. If \( f(z) \in F \) then \( |f'(z)| < 2 \). From \( f(z_2) - f(z_1) = \int_{z_1}^{z_2} f'(z) \, dz \) we obtain \(|f(z_2) - f(z_1)| \leq 2 |z_2 - z_1|\). This implies that \( f(z) \) is uniformly continuous in \(|z| < 1\) and consequently can be extended continuously onto \(|z| = 1\). Let \( C \) be defined by \( w = f(e^{i\theta}) \), \( 0 \leq \theta \leq 2\pi \).

Let us prove that \( C \) is rectifiable. This follows easily from the estimate \(|f(z_2) - f(z_1)| \leq 2 |z_2 - z_1|\) for \(|z_1| = |z_2| = 1\). Namely, if \( 0 = \theta_0 < \theta_1 < \theta_2 < \cdots < \theta_n = 2\pi \) then

\[
\sum_{k=1}^{n} |f(e^{i\theta_k}) - f(e^{i\theta_{k-1}})| \leq 2 \sum_{k=1}^{n} |e^{i\theta_k} - e^{i\theta_{k-1}}| < 4\pi.
\]

Next we show that each function \( f(z) \) in \( F \) is univalent in \(|z| \leq 1\). This follows from the facts: \( \Re f'(z) > 0 \) for \(|z| < 1\) and \( f(z) \) is continuous in \(|z| \leq 1\). It suffices to prove that \( f(z) \) is univalent on \(|z| = 1\). Suppose that \( z_1 \neq z_2 \), \(|z_1| = |z_2| = 1\). Let \( l \) denote the line segment from \( z_1 \) to \( z_2 \). We will consider several points, denoted by \( z_3, z_4, z_5, z_6, z_7, z_8 \). These will be distinct points on \( l \) and arranged in the order as written. Fix \( z_4 \) and then choose \( z_3 \) and \( z_6 \) such that \( \Re f'(z) \leq (1/2) \Re f'(z_4) \) for all points on \( l \) between \( z_3 \) and \( z_6 \). Then, integrating along \( l \) we obtain

\[
|f(z_3) - f(z_1)| = \int_{z_1}^{z_3} f'(z) \, dz
\]

\[
\geq (1/2) |z_3 - z_1| \Re f'(z_4) = a.
\]

With \( z_3, z_4, z_5 \) fixed we can choose \( z_1 \) and \( z_2 \) so close to \( z_1 \) and \( z_2 \) that

\[
|f(z_2) - f(z_1) - (f(z_1) - f(z_1))| < a.
\]

Therefore, \( f(z_2) - f(z_1) \neq 0 \). This proves that \( f(z) \) is univalent in \(|z| \leq 1\). In particular this shows that \( C \) is simple.

Theorem 4. Suppose \( 0 < r < 1 \). The length of the image of \(|z| = r\) under functions in \( F \) is maximal for \( f_0(z) = z + (1/2)z^2 \). Moreover, this length is less than 8.

Proof. Suppose that \( f(z) \in F \) and \( L_r(f) \) is the length of the image of \(|z| = r\) under \( f(z) \). Since \(|f'(z) - 1| < 1\) and \( f''(z) - 1 \) vanishes at \( z = 0 \), \( f'(z) \) is subordinate to \( f_0'(z) = 1 + z \) in \(|z| < 1\). J. E. Littlewood [2, p. 484, Theorem 2] has shown that if \( h(z) \) is subordinate to \( H(z) \) in \(|z| < 1\) then

\[
\int_{0}^{2\pi} |h(re^{i\theta})|^2 \, d\theta \leq \int_{0}^{2\pi} |H(re^{i\theta})|^2 \, d\theta,
\]
for any $k > 0$. Applying this result to $f'(z)$ for $k = 1$ we prove one part of the theorem, as follows.

$$L_r(f) = r \int_0^{2\pi} |f'(re^{i\theta})| \, d\theta \leq r \int_0^{2\pi} |f''(re^{i\theta})| \, d\theta = L_r(f_0).$$

$L_r(f) < 8$ will hold if we prove $L_r(f_0) < 8$. If $R \geq 1$ then $f'(z) = 1 + z$ is subordinate to $w = 1 + Rz$ in $|z| < 1$. With $R = 1/r$ we again apply Littlewood’s inequality.

$$L_r(f_0) < \int_0^{2\pi} |f'(re^{i\theta})| \, d\theta$$

$$\leq \int_0^{2\pi} |1 + e^{i\theta}| \, d\theta$$

$$= 2 \int_0^{2\pi} |\cos(\theta/2)| \, d\theta$$

$$= 8.$$

Remarks. 1. It seems likely that the length of the boundary of the image of $|z| < 1$ under each function in $F$ satisfies $L \leq 8$. This could not be improved since $L = 8$ for the functions $f(z) = z + a_n z^n$, $|a_n| = 1/n$, $n = 2, 3, \ldots$.

2. Littlewood’s inequality for $k = 2$ gives another proof of the estimate $A \leq (3/2)\pi$ in Theorem 2.

5. Radii of convexity and starlikeness.

Theorem 5. Each function in $F$ maps $|z| < 1/2$ onto a convex domain.

Proof. Suppose that $f(z) \in F$. Then $f'(z) = 1 + zg(z)$, where $g(z)$ is regular for $|z| < 1$ and $|g(z)| \leq 1$. For such functions we have the estimate

$$|g'(z)| \leq \frac{1 - |g(z)|^2}{1 - |z|^2}$$

[4, p. 168]. Also,

$$\frac{f''(z)}{f'(z)} = \frac{g(z) + zg'(z)}{1 + zg(z)}.$$

Using the triangle inequalities and then the estimate for $|g'(z)|$ we find that

$$\left|\frac{f''(z)}{f'(z)}\right| \leq \frac{|g(z)| + |z|}{1 - |z|^2}.$$
Since \(|g(z)| \leq 1\) we obtain
\[
\left| \frac{f''(z)}{f'(z)} \right| \leq \frac{1}{1 - |z|}.
\]

The condition \(\text{Re}\left\{zf''(z)/f'(z) + 1\right\} > 0\) for \(|z| < r\) is necessary and sufficient for \(f(z)\) to map \(|z| < r\) onto a convex domain. This condition is satisfied if \(|zf''(z)/f'(z)| < 1\). Thus, \(f(z)\) is convex if \(|z|/(1 - |z|) < 1\), i.e., if \(|z| < 1/2\).

It is not difficult to see that \(f(z) = z + a_2z^2, \ |a_2| = 1/2\), are the only functions in \(F\) which are not convex in \(|z| < r\) for some \(r > 1/2\).

**Theorem 6.** Each function in \(F\) maps \(|z| < (4/5)^{1/2}\) onto a domain which is starlike with respect to the origin.

**Proof.** The condition \(\text{Re}\left\{zf''(z)/f'(z)\right\} > 0\) for \(|z| < r\) is necessary and sufficient for \(f(z)\) to be starlike in \(|z| < r\). In §2 we proved that \(|f'(z) - 1| \leq |z|\) and \(|f(z) - z| \leq (1/2)|z|^2\) if \(f(z) \in F\). Taking advantage of the geometric location of \(f'(z)\) and \(f(z)/z\) as given by these inequalities we obtain
\[
\left| \arg f'(z) \right| \leq \sin^{-1} |z|, \quad \left| \arg \frac{f(z)}{z} \right| \leq \sin^{-1} (|z|/2).
\]

If \(|z| < (4/5)^{1/2}\) then \(\sin^{-1}|z| + \sin^{-1}(|z|/2) < \pi/2\). Therefore, for \(|z| < (4/5)^{1/2}\) we have
\[
\left| \arg \frac{zf'(z)}{f(z)} \right| \leq \left| \arg f'(z) \right| + \left| \arg \frac{f(z)}{z} \right| < \pi/2,
\]
i.e., \(\text{Re}\left\{zf''(z)/f'(z)\right\} > 0\) for \(|z| < (4/5)^{1/2}\).

The estimates used in this proof are precise only for the functions \(f(z) = z + a_2z^2\), where \(|a_2| = 1/2\). Since these functions map the whole circle \(|z| < 1\) onto a starlike domain, \((4/5)^{1/2}\) is not the radius of starlikeness for the class \(F\).

**6. Functions with initial zero coefficients.** Some of the results obtained for functions in \(F\) can be improved if \(f(z)\) has the form \(f(z) = z + a_nz^n + a_{n+1}z^{n+1} + \cdots\). In this situation we can write \(f'(z) - 1 = z^{n-1}g(z)\), where \(g(z)\) is regular for \(|z| < 1\) and \(|g(z)| \leq 1\). With this as the starting point we can argue as in §§2 and 3 to prove the following theorem. The extremal functions for this theorem are \(f(z) = z + a_nz^n\), where \(|a_n| = 1/n\).
Theorem 7. Suppose that \( f(z) = z + a_1z^n + a_{n+1}z^{n+1} + \cdots \in F \). Then 
\( f(z) \) assumes all values in the circle \( |w| < 1 - (1/n) \). No values outside of the circle \( |w| < 1 + (1/n) \) are assumed. The area of the image domain satisfies \( A \leq \pi (1 + 1/n) \).

Theorem 8. If \( f(z) = z + a_1z^n + a_{n+1}z^{n+1} + \cdots \in F \) then \( f(z) \) maps \( |z| < (1/n)^{1/(n-1)} \) onto a convex domain.

Proof. From \( f'(z) - 1 = z^{n-1}g(z) \) it follows that

\[
\frac{f''(z)}{f'(z)} = z^{n-2} \frac{(n-1)g(z) + zg'(z)}{1 + z^{n-1}g(z)}.
\]

Using the triangle inequalities and then the estimate \( |g'(z)| \leq (1 - |g(z)|^2)/(1 - |z|^2) \), we obtain

\[
\left| \frac{f''(z)}{f'(z)} \right| \leq \frac{|z|^{n-2}}{1 - |z|^2} \left| z + (n-1)(1 - |z|^2)g(z) \right| \left| 1 - |z|g(z) \right|^2.
\]

To estimate the right side of this inequality let us consider the function

\[
y = a + (n-1)(1-a^2)x - ax^2,
\]

\[
a = |z|, \quad x = |g(z)|, \quad 0 < a < 1, \quad 0 \leq x \leq 1.
\]

\[
p = (1-a^{n-1}x)^2 \frac{dy}{dx} = (n-1)(1-a^2) + a^n - 2ax + anx^2
\]

\[
\frac{dp}{dx} = 2a(a^{n-1}x - 1) < 0.
\]

Thus \( p \) is decreasing.

\[
p(x) \geq p(1) = (n-1)(1-a^2) - 2a(1-a^{n-1})
\]

\[
= (1-a)[(n-1)(1+a) - 2a(1+a + a^2 + \cdots + a^{n-2})]
\]

\[
> (1-a)[(n-1)(1+a) - 2a(n-1)]
\]

\[
= (n-1)(1-a)^2 > 0.
\]

Thus \( y \) is an increasing function. Therefore

\[
y(x) \leq y(1) = \frac{(n-1)(1-a^2)}{1-a^{n-1}},
\]

\[
\left| \frac{f''(z)}{f'(z)} \right| \leq \frac{(n-1)|z|^{n-2}}{1 - |z|^{n-1}}.
\]
If \(|zf''(z)/f'(z)| < 1\) then \(f(z)\) is convex. Thus, \(f(z)\) is convex if 
\[ (n-1)|z|^{n-1}/(1-|z|^{n-1}) < 1, \]
and if \(|z| < (1/n)^{1/(n-1)}\).

The functions \(f(z)=z+a_nz^n, \ |a_n|=1/n,\) are extremal, i.e., they 
are the only functions in \(F\) of the form \(f(z)=z+a_nz^n+a_{n+1}z^{n+1}+\cdots\)
which are not convex in some circle \(|z| < r, r > (1/n)^{1/(n-1)}\).

References


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