A CLASS OF UNIVALENT FUNCTIONS[1,2]

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1. **Introduction.** The condition $\operatorname{Re} f'(z)>0$ is known to be sufficient for the univalence of an analytic function in any convex domain. In a recent paper [3] the author investigated the class of functions which satisfy $\operatorname{Re} f'(z)>0$ for $|z|<1$ and are normalized by $f(0)=0$, $f'(0)=1$. In this paper we study the subclass denoted by F and defined by the condition $|f'(z)-1|<1$ for $|z|<1$. Some of our results have already been proven for the particular functions in F whose coefficients satisfy $\sum_{n=2}^{\infty} n|a_n| \leqq 1$. Particular reference should be made of a paper by Schild [5].

2. **Distortion theorems.** Suppose that $f(z)\in F$. By applying Schwarz's lemma to the function $f'(z)-1$ we obtain $|f'(z)-1| \leqq |z|$. This gives the estimates $1-|z| \leqq |f'(z)| \leqq 1+|z|$. Bounds for $|f(z)|$ can be obtained from $|f'(z)-1| \leqq |z|$ by integration, as follows. Integrating along the line segment from 0 to z we may write

$$f(z) - z = \int_0^z (f'(s) - 1)ds = z\int_0^1 (f'(tz) - 1)dt.$$

$$\begin{aligned} |f(z) - z| &\leqq |z| \int_0^1 |f'(tz) - 1|\, dt \\ &\leqq |z| \int_0^1 t|z|\, dt = (1/2)|z|^2. \end{aligned}$$

From this estimate for $|f(z)-z|$ we immediately obtain

$$|z| - (1/2)|z|^2 \leqq |f(z)| \leqq |z| + (1/2)|z|^2.$$

Each estimate is precise only for the functions $f(z)=z+a_2z^2$, where $|a_2|=1/2$.

The bounds for $|f(z)|$ imply the following theorem.

THEOREM 1. *Each function in F assumes every complex number in the circle $|w|<1/2$. No values outside of the circle $|w|<3/2$ are assumed.*

Received by the editors January 22, 1963.

[1] Most of the results in this paper are contained in the author's dissertation, which was completed at the University of Pennsylvania in June, 1961.

[2] The preparation of this paper was partially supported by the National Science Foundation under the grant NSF-GP-161.

3. **An area theorem.** If $g(z)=\sum_{n=0}^{\infty} b_n z^n$ is regular for $|z|<1$ and $|g(z)|\leqq 1$ then $\sum_{n=0}^{\infty}|b_n|^2\leqq 1$ [1 p. 7]. Applying this estimate to $f'(z)-1$ where $f(z)=z+\sum_{n=2}^{\infty} a_n z^n\in F$ we get $\sum_{n=2}^{\infty} n^2|a_n|^2\leqq 1$. We will use this estimate in the next theorem. It also shows that the coefficients of functions in F satisfy $|a_n|\leqq 1/n$ for $n=2, 3, \cdots$ and equality for a given n holds only for functions of the form $f(z)=z+a_n z^n$.

THEOREM 2. *The area of the image of* $|z|<1$ *under each function in* F *satisfies* $A\leqq(3/2)\pi$.

PROOF. Suppose that $f(z)=z+\sum_{n=2}^{\infty} a_n z^n\in F$ and $0<r<1$. Let D_r and D be the images of $|z|<r$ and $|z|<1$ under $f(z)$ and let A_r and A be the areas of D_r and D. The area of D exists since D is open and bounded. A_r is given by the following well-known formula.

$$A_r = \pi\left\{r^2+\sum_{n=2}^{\infty} n|a_n|^2 r^{2n}\right\}.$$

We will prove that

$$A = \pi\left\{1+\sum_{n=2}^{\infty} n|a_n|^2\right\}. \tag{1}$$

The convergence of this series follows from the fact that $\sum_{n=2}^{\infty} n^2|a_n|^2$ converges. Let $\{r_n\}$ be any increasing sequence of positive numbers such that $r_n\to 1$. Then $\{D_{r_n}\}$ is an increasing sequence of sets whose union is D. From the theorem in measure theory $A_{r_n}\to A$. However the convergence of (1) implies that A_r is continuous for $0\leqq r\leqq 1$. Consequently $A_{r_n}\to A_1$. This proves (1).

From (1) and $\sum_{n=2}^{\infty} n^2|a_n|^2\leqq 1$, we obtain

$$\begin{aligned} A &= \pi\left\{1+\sum_{n=2}^{\infty} n|a_n|^2\right\} \\ &\leqq \pi\left\{1+(1/2)\sum_{n=2}^{\infty} n^2|a_n|^2\right\} \\ &\leqq (3/2)\pi. \end{aligned}$$

This proves $A\leqq(3/2)\pi$ and $A=(3/2)\pi$ only for the functions $f(z)=z+a_2z^2$ where $|a_2|=1/2$.

4. **The boundary of the image domain.**

THEOREM 3. *Each function in* F *maps* $|z|<1$ *onto a domain whose boundary is a rectifiable Jordan curve.*

PROOF. If $f(z)\in F$ then $|f'(z)|<2$. From $f(z_2)-f(z_1)=\int_{z_1}^{z_2}f'(z)dz$ we obtain $|f(z_2)-f(z_1)|\leqq 2|z_2-z_1|$. This implies that $f(z)$ is uniformly continuous in $|z|<1$ and consequently can be extended continuously onto $|z|=1$. Let C be defined by $w=f(e^{i\theta})$, $0\leqq\theta\leqq 2\pi$.

Let us prove that C is rectifiable. This follows easily from the estimate $|f(z_2)-f(z_1)|\leqq 2|z_2-z_1|$ for $|z_1|=|z_2|=1$. Namely, if $0=\theta_0<\theta_1<\theta_2<\cdots<\theta_n=2\pi$ then

$$\sum_{k=1}^{n}|f(e^{i\theta_k})-f(e^{i\theta_{k-1}})|\leqq 2\sum_{k=1}^{n}|e^{i\theta_k}-e^{i\theta_{k-1}}|<4\pi.$$

Next we show that each function $f(z)$ in F is univalent in $|z|\leqq 1$. This follows from the facts: $\operatorname{Re} f'(z)>0$ for $|z|<1$ and $f(z)$ is continuous in $|z|\leqq 1$. It suffices to prove that $f(z)$ is univalent on $|z|=1$. Suppose that $z_1\neq z_2$, $|z_1|=|z_2|=1$. Let l denote the line segment from z_1 to z_2. We will consider several points, denoted by z_1, s_1, z_3, z_4, z_5, s_2, z_2. These will be distinct points on l and arranged in the order as written. Fix z_4 and then choose z_3 and z_5 such that $\operatorname{Re} f'(z)\geqq(1/2)\operatorname{Re} f'(z_4)$ for all points on l between z_3 and z_5. Then, integrating along l we obtain

$$\begin{aligned}|f(s_2)-f(s_1)| &= \left|\int_{s_1}^{s_2}f'(z)dz\right| \\ &\geqq (1/2)|z_5-z_3|\ \operatorname{Re} f'(z_4)=a.\end{aligned}$$

With z_3, z_4, z_5 fixed we can choose s_1 and s_2 so close to z_1 and z_2 that

$$|f(z_2)-f(s_2)-(f(z_1)-f(s_1))|<a.$$

Therefore, $f(z_2)-f(z_1)\neq 0$. This proves that $f(z)$ is univalent in $|z|\leqq 1$. In particular this shows that C is simple.

THEOREM 4. *Suppose* $0<r<1$. *The length of the image of* $|z|=r$ *under functions in* F *is maximal for* $f_0(z)=z+(1/2)z^2$. *Moreover, this length is less than* 8.

PROOF. Suppose that $f(z)\in F$ and $L_r(f)$ is the length of the image of $|z|=r$ under $f(z)$. Since $|f'(z)-1|<1$ and $f'(z)-1$ vanishes at $z=0$, $f'(z)$ is subordinate to $f_0'(z)=1+z$ in $|z|<1$. J. E. Littlewood [2, p. 484, Theorem 2] has shown that if $h(z)$ is subordinate to $H(z)$ in $|z|<1$ then

$$\int_0^{2\pi}|h(re^{i\theta})|^k d\theta\leqq\int_0^{2\pi}|H(re^{i\theta})|^k d\theta,$$

for any $k>0$. Applying this result to $f'(z)$ for $k=1$ we prove one part of the theorem, as follows.

$$L_r(f) = r\int_0^{2\pi} |f'(re^{i\theta})|\, d\theta \leqq r\int_0^{2\pi} |f_0'(re^{i\theta})|\, d\theta = L_r(f_0).$$

$L_r(f)<8$ will hold if we prove $L_r(f_0)<8$. If $R\geqq 1$ then $f_0'(z)=1+z$ is subordinate to $w=1+Rz$ in $|z|<1$. With $R=1/r$ we again apply Littlewood's inequality.

$$\begin{aligned} L_r(f_0) &< \int_0^{2\pi} |f_0'(re^{i\theta})|\, d\theta \\ &\leqq \int_0^{2\pi} |1+e^{i\theta}|\, d\theta \\ &= 2\int_0^{2\pi} |\cos(\theta/2)|\, d\theta \\ &= 8. \end{aligned}$$

REMARKS. 1. It seems likely that the length of the boundary of the image of $|z|<1$ under each function in F satisfies $L\leqq 8$. This could not be improved since $L=8$ for the functions $f(z)=z+a_nz^n$, $|a_n|=1/n$, $n=2, 3, \cdots$.

2. Littlewood's inequality for $k=2$ gives another proof of the estimate $A\leqq(3/2)\pi$ in Theorem 2.

5. Radii of convexity and starlikeness.

THEOREM 5. *Each function in F maps $|z|<1/2$ onto a convex domain.*

PROOF. Suppose that $f(z)\in F$. Then $f'(z)-1=zg(z)$, where $g(z)$ is regular for $|z|<1$ and $|g(z)|\leqq 1$. For such functions we have the estimate

$$|g'(z)| \leqq \frac{1-|g(z)|^2}{1-|z|^2}$$

[4, p. 168]. Also,

$$\frac{f''(z)}{f'(z)} = \frac{g(z)+zg'(z)}{1+zg(z)}.$$

Using the triangle inequalities and then the estimate for $|g'(z)|$ we find that

$$\left|\frac{f''(z)}{f'(z)}\right| \leqq \frac{|g(z)|+|z|}{1-|z|^2}.$$

Since $|g(z)| \leqq 1$ we obtain

$$\left|\frac{f''(z)}{f'(z)}\right| \leqq \frac{1}{1-|z|}.$$

The condition $\operatorname{Re}\{zf''(z)/f'(z)+1\}>0$ for $|z|<r$ is necessary and sufficient for $f(z)$ to map $|z|<r$ onto a convex domain. This condition is satisfied if $|zf''(z)/f'(z)|<1$. Thus, $f(z)$ is convex if $|z|/(1-|z|)<1$, i.e., if $|z|<1/2$.

It is not difficult to see that $f(z)=z+a_2z^2$, $|a_2|=1/2$, are the only functions in F which are not convex in $|z|<r$ for some $r>1/2$.

THEOREM 6. *Each function in F maps $|z|<(4/5)^{1/2}$ onto a domain which is starlike with respect to the origin.*

PROOF. The condition $\operatorname{Re}\{zf'(z)/f(z)\}>0$ for $|z|<r$ is necessary and sufficient for $f(z)$ to be starlike in $|z|<r$. In §2 we proved that $|f'(z)-1| \leqq |z|$ and $|f(z)-z| \leqq (1/2)|z|^2$ if $f(z)\in F$. Taking advantage of the geometric location of $f'(z)$ and $f(z)/z$ as given by these inequalities we obtain

$$|\arg f'(z)| \leqq \sin^{-1}|z|,$$

$$\left|\arg\frac{f(z)}{z}\right| \leqq \sin^{-1}(|z|/2).$$

If $|z|<(4/5)^{1/2}$ then $\sin^{-1}|z|+\sin^{-1}(|z|/2)<\pi/2$. Therefore, for $|z|<(4/5)^{1/2}$ we have

$$\left|\arg\frac{zf'(z)}{f(z)}\right| \leqq |\arg f'(z)| + \left|\arg\frac{f(z)}{z}\right| < \pi/2,$$

i.e., $\operatorname{Re}\{zf'(z)/f(z)\}>0$ for $|z|<(4/5)^{1/2}$.

The estimates used in this proof are precise only for the functions $f(z)=z+a_2z^2$, where $|a_2|=1/2$. Since these functions map the whole circle $|z|<1$ onto a starlike domain, $(4/5)^{1/2}$ is not the radius of starlikeness for the class F.

6. **Functions with initial zero coefficients.** Some of the results obtained for functions in F can be improved if $f(z)$ has the form $f(z)=z+a_nz^n+a_{n+1}z^{n+1}+\cdots$. In this situation we can write $f'(z)-1=z^{n-1}g(z)$, where $g(z)$ is regular for $|z|<1$ and $|g(z)|\leqq 1$. With this as the starting point we can argue as in §§2 and 3 to prove the following theorem. The extremal functions for this theorem are $f(z)=z+a_nz^n$, where $|a_n|=1/n$.

THEOREM 7. *Suppose that* $f(z)=z+a_nz^n+a_{n+1}z^{n+1}+\cdots\in F$. *Then* $f(z)$ *assumes all values in the circle* $|w|<1-(1/n)$. *No values outside of the circle* $|w|<1+(1/n)$ *are assumed. The area of the image domain satisfies* $A\leqq\pi(1+1/n)$.

THEOREM 8. *If* $f(z)=z+a_nz^n+a_{n+1}z^{n+1}+\cdots\in F$ *then* $f(z)$ *maps* $|z|<(1/n)^{1/(n-1)}$ *onto a convex domain.*

PROOF. From $f'(z)-1=z^{n-1}g(z)$ it follows that

$$\frac{f''(z)}{f'(z)}=z^{n-2}\frac{(n-1)g(z)+zg'(z)}{1+z^{n-1}g(z)}.$$

Using the triangle inequalities and then the estimate $|g'(z)|\leqq(1-|g(z)|^2)/(1-|z|^2)$, we obtain

$$\left|\frac{f''(z)}{f'(z)}\right|\leqq\frac{|z|^{n-2}}{1-|z|^2}\,\frac{|z|+(n-1)(1-|z|^2)|g(z)|-|z|\,|g(z)|^2}{1-|z|^{n-1}|g(z)|}.$$

To estimate the right side of this inequality let us consider the function

$$y=\frac{a+(n-1)(1-a^2)x-ax^2}{1-a^{n-1}x},$$

$$a=|z|,\quad x=|g(z)|,\quad 0<a<1,\quad 0\leqq x\leqq 1.$$

$$p=(1-a^{n-1}x)^2\frac{dy}{dx}=(n-1)(1-a^2)+a^n-2ax+a^nx^2$$

$$\frac{dp}{dx}=2a(a^{n-1}x-1)<0.$$

Thus p is decreasing.

$$\begin{aligned}p(x)\geqq p(1)&=(n-1)(1-a^2)-2a(1-a^{n-1})\\&=(1-a)[(n-1)(1+a)-2a(1+a+a^2+\cdots+a^{n-2})]\\&>(1-a)[(n-1)(1+a)-2a(n-1)]\\&=(n-1)(1-a)^2>0.\end{aligned}$$

Thus y is an increasing function. Therefore

$$y(x)\leqq y(1)=\frac{(n-1)(1-a^2)}{1-a^{n-1}},$$

$$\left|\frac{f''(z)}{f'(z)}\right|\leqq\frac{(n-1)|z|^{n-2}}{1-|z|^{n-1}}.$$

If $|zf''(z)/f'(z)|<1$ then $f(z)$ is convex. Thus, $f(z)$ is convex if $(n-1)|z|^{n-1}/(1-|z|^{n-1})<1$, i.e., if $|z|<(1/n)^{1/(n-1)}$.

The functions $f(z)=z+a_nz^n$, $|a_n|=1/n$, are extremal, i.e., they are the only functions in F of the form $f(z)=z+a_nz^n+a_{n+1}z^{n+1}+\cdots$ which are not convex in some circle $|z|<r$, $r>(1/n)^{1/(n-1)}$.

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