CONTRACTION OF WALSH FOURIER SERIES

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The purpose of this note is to prove an analogue for Walsh Fourier series (abbreviated WFS) of a theorem of R. P. Boas [1]. For basic properties of Walsh functions, the reader is referred to N. J. Fine [2].

We begin with some notations and definitions:

The Walsh functions are denoted by \( \psi_n(x) \) \((n = 0, 1, 2, \ldots)\) and considered in the “real version.”

A complex-valued function \( T(z) \) of a complex variable \( z \) is called a contraction if it satisfies the inequality

\[ |T(z_1) - T(z_2)| \leq |z_1 - z_2|. \]

An integrable function \( f(x) \) is contractible if it has absolutely convergent WFS and the same is true for \( T(f(x)) \) where \( T \) is any contraction.

Our theorem now reads as follows:

**Theorem.** Let \( \{\omega_n\} \) be a sequence of non-negative numbers with \( \sum_{n=1}^{\infty} \omega_n < \infty \) and

\[
\sum_{n=1}^{\infty} n^{-3/2} \left( \sum_{\nu=1}^{n} 2^{\nu/2} \omega_{\nu} \right)^{1/2} + \sum_{n=1}^{\infty} n^{-1/2} \left( \sum_{\nu=1}^{n} 2^{\nu/2} \omega_{\nu} \right)^{1/2} < \infty.
\]

Then an integrable function \( f(x) \) with

\[
f(x) \sim \sum_{n=0}^{\infty} c_n \psi_n(x), \quad |c_n| \leq \omega_n
\]

is contractible.

The proof of this theorem is essentially the same as Boas’, the only difference being Lemma 3 below, in which an elementary property of Walsh function is used.

**Lemma 1.** For a non-negative sequence \( \{\omega_n\} \), (1) is equivalent to

\[
\sum_{n=1}^{\infty} 2^{-n/2} \left( \sum_{\nu=1}^{2^{n-1}} 2^{\nu/2} \omega_{\nu} \right)^{1/2} + \sum_{n=1}^{\infty} 2^{n/2} \left( \sum_{\nu=2^n}^{2^{n+1}} \omega_{\nu} \right)^{1/2} < \infty
\]

where \( k = k(\nu) \) is the integer satisfying \( 2^k \leq \nu < 2^{k+1} \).

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\(^1\) We consider measurable functions with period 1 only; thus the integrability is meant over an interval of length 1, say \([0, 1)\).
PROOF. The equivalence between
\[ \sum_{n=1}^{\infty} n^{-1/2} \left( \sum_{r=n}^{\infty} \frac{1}{\omega_r} \right)^{1/2} < \infty \quad \text{and} \quad \sum_{n=1}^{\infty} 2^{n/2} \left( \sum_{r=n}^{\infty} \frac{1}{\omega_r} \right)^{1/2} < \infty \]
is nothing but Cauchy's condensation theorem. On the other hand,
\[
\sum_{n=1}^{\infty} n^{-3/2} \left( \sum_{r=1}^{n} \frac{1}{\omega_r} \right)^{1/2}
\]
\[= \sum_{j=0}^{\infty} \sum_{n=2^j}^{2^{j+1}-1} n^{-3/2} \left( \sum_{r=1}^{n} \frac{1}{\omega_r} \right)^{1/2} \leq \sum_{j=0}^{\infty} 2^{-3j/2} \sum_{n=2^j}^{2^{j+1}-1} \left( \sum_{r=1}^{n} \frac{1}{\omega_r} \right)^{1/2} \]
\[\leq A \sum_{j=0}^{\infty} 2^{-3j/2} \left( \sum_{r=1}^{2^{j+1}-1} 2^{k-1} \omega_r \right)^{1/2} = A \sum_{j=0}^{\infty} 2^{-j/2} \left( \sum_{r=1}^{2^{j+1}-1} 2^{k-1} \omega_r \right)^{1/2},
\]
and
\[
\sum_{j=0}^{\infty} \sum_{n=2^j}^{2^{j+1}-1} n^{-3/2} \left( \sum_{r=1}^{n} \frac{1}{\omega_r} \right)^{1/2} \geq \sum_{j=0}^{\infty} \sum_{n=2^j}^{2^{j+1}-1} 2^{-3j/2} \left( \sum_{r=1}^{n} \frac{1}{\omega_r} \right)^{1/2} \geq A \sum_{j=0}^{\infty} 2^{-3j/2} \left( \sum_{n=2^j}^{2^{j+1}-1} 2^{k-1} \omega_r \right)^{1/2} = A \sum_{j=0}^{\infty} 2^{-j/2} \left( \sum_{r=1}^{2^{j+1}-1} 2^{k-1} \omega_r \right)^{1/2}, \quad \text{q.e.d.}
\]

**Lemma 2.** For any non-negative sequence \( \{a_r\} \),
\[
\sum_{r=1}^{\infty} a_r \leq \sum_{n=1}^{\infty} 2^{-n/2} \left( \sum_{r=1}^{2^n-1} 2^{k-1} a_r \right)^{1/2}.
\]

**Proof.**
\[
\sum_{r=1}^{\infty} a_r = \sum_{j=0}^{\infty} \sum_{r=2^j}^{2^{j+1}-1} a_r = \sum_{j=0}^{\infty} 2^j \sum_{r=2^j}^{2^{j+1}-1} a_r = \sum_{j=0}^{\infty} 2^j \sum_{r=2^j}^{2^{j+1}-1} 2^{k-1} a_r
\]
\[= \sum_{n=1}^{\infty} 2^{-n} \sum_{r=0}^{2^n-1} 2^{k-1} a_r \leq \sum_{n=1}^{\infty} 2^{-n} \left( \sum_{r=1}^{2^n-1} 2^{k-1} a_r \right)^{1/2} \]
\[= \sum_{n=1}^{\infty} 2^{-n/2} \left( \sum_{r=1}^{2^n-1} 2^{k-1} a_r \right)^{1/2}, \quad \text{q.e.d}
\]
**Lemma 3.** Let \( f(x) = \sum_{r=0}^{\infty} c_r \psi_r(x) \) with \( \sum_{r=0}^{\infty} |c_r|^2 < \infty \) and let \( g(x) \sim \sum_{r=0}^{\infty} a_r \psi_r(x) \) be a contraction of \( f(x) \). Then

\[
\sum_{r=1}^{2^n-1} 2^{2k} |a_r|^2 \leq A \sum_{r=1}^{2^n-1} 2^{2k} |c_r|^2 + A \cdot 2^n \sum_{r=2^n}^\infty |c_r|^2.
\]

**Proof.** For any \( h, 0 < h < 1 \), choose the integer \( m \) so that

\[2^{-m-1} \leq h < 2^{-m}.
\]

We have, by the definition of Walsh functions,

\[\psi_r(h) = -1 \quad (2^m \leq r < 2^{m+1}).\]

From Parseval’s relation combined with (2), it follows that

\[
4 \sum_{r=2^m}^{2^{m+1}-1} |a_r|^2 \leq \sum_{r=2^m}^{\infty} |a_r|^2 \left| 1 - \psi_r(h) \right|^2 = \int_0^1 |g(x + h) - g(x)|^2 \, dx
\]

\[
\leq \int_0^1 |f(x + h) - f(x)|^2 \, dx = \sum_{r=2^m}^{\infty} |c_r|^2 \left| 1 - \psi_r(h) \right|^2
\]

\[
\leq 4 \sum_{r=2^m}^{\infty} |c_r|^2.
\]

Thus we have

\[
\sum_{r=2^m}^{2^{m+1}-1} |a_r|^2 \leq \sum_{r=2^m}^{\infty} |c_r|^2 = \sum_{j=m}^{\infty} \sum_{r=2^j}^{2^{j+1}-1} |c_r|^2.
\]

Since (3) is true for each \( m \), multiplication by \( 2^{2m} \) and summation with respect to \( m \) give

\[
\sum_{m=0}^{n-1} 2^{2m} \sum_{r=2^m}^{2^{m+1}-1} |a_r|^2 = \sum_{m=0}^{n-1} 2^{2m} \sum_{r=2^m}^{2^{m+1}-1} 2^{2k} |a_r|^2 \leq \sum_{m=0}^{n-1} 2^{2m} \sum_{j=m}^{\infty} \sum_{r=2^j}^{2^{j+1}-1} |c_r|^2
\]

\[
= \sum_{j=0}^{n-1} \sum_{r=2^j}^{2^{j+1}-1} |c_r|^2 2^{2j} + \sum_{j=m}^{\infty} \sum_{r=2^j}^{2^{j+1}-1} |c_r|^2 2^{2m} = \sum_{j=m}^{n-1} 2^{2j} |c_r|^2 + \sum_{j=m}^{\infty} 2^{2m} |c_r|^2
\]

\[
\leq A \sum_{j=0}^{n-1} 2^{2j} |c_r|^2 + A \cdot 2^{2n} \sum_{j=m}^{\infty} |c_r|^2
\]

\[
\leq A \sum_{r=1}^{2^n-1} |c_r|^2 + A \cdot 2^{2n} \sum_{r=2^n}^\infty |c_r|^2, \quad \text{q.e.d.}
\]
Proof of Theorem. By Lemmas 2 and 3, we have

\[ \sum_{r=1}^{\infty} |a_r| \leq A \sum_{n=1}^{\infty} 2^{-n/2} \left( \sum_{r=1}^{2^{n-1}} |a_r|^2 \right)^{1/2} \]
\[ \leq A \sum_{n=1}^{\infty} 2^{-n/2} \left( \sum_{r=1}^{2^{n-1}} 2^{2k} |c_r|^2 + 2^{2n} \sum_{r=2^n}^{\infty} |c_r|^2 \right)^{1/2} \]
\[ \leq A \sum_{n=1}^{\infty} 2^{-n/2} \left( \sum_{r=1}^{2^{n-1}} 2^{2k} \omega_r \right)^{1/2} + A \sum_{n=1}^{\infty} 2^n |\omega_r| \left( \sum_{r=2^n}^{\infty} \omega_r \right)^{1/2}, \]

which is convergent by Lemma 1 and the assumption of Theorem.

References


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