2(2d+1) is not a kth power residue modulo p. Since n was arbitrary then $\Lambda(k, 4) = \infty$. This proves the theorem.

References


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ON DECOMPOSITIONS OF PARTIALLY ORDERED SETS

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1. Introduction. Let P be a set which is partially ordered by a relation $\leq$. A decomposition $\mathcal{D}$ of P is a family of mutually disjoint non-empty chains in P such that $P = \bigcup \{ C : C \in \mathcal{D} \}$. Two elements $x, y$ of P are incomparable if and only if $x \not\leq y$ and $y \not\leq x$. A totally unordered set in P is a subset in which every two different elements are incomparable. We denote the cardinal number of a set $S$ by $|S|$.

Dilworth [1] has proved the following well-known decomposition theorem.

Theorem 1 (Dilworth). Let P be a partially ordered set, and suppose that $n$ is a positive integer such that

$$n = \max \{|A| : A \text{ is a totally unordered subset of } P\}.$$

Then there is a decomposition $\mathcal{D}$ of P with $|\mathcal{D}| = n$.

It is natural to ask whether, in this theorem, the positive integer $n$ may be replaced by an infinite cardinal number. However, the theorem is no longer valid in this case, as is shown by an example in [3] which is due in essence of Sierpinski [2]. In this example $P$ is a set of pairs which represents a 1-1 mapping from $\omega_1$, the first uncountable ordinal, into the real numbers. $(x_1, y_1) \leq (x_2, y_2)$ is defined by: $x_1 \leq x_2$ (as ordinals) and $y_1 \leq y_2$ (as real numbers). The purpose of this note is to show that a similar idea leads, given any infinite cardinal $k$, to

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a construction of a partially ordered set in which all totally unordered subsets are finite but every decomposition is of power \( k \). We also give an application of this result to the theory of graphs.

In the following we identify cardinals with initial ordinals. If \( C \) is any chain and \( B \subseteq C \), we shall say that \( B \) is cofinal in \( C \) if and only if for each \( x \in C \) there exists \( y \in B \) with \( x \leq y \).

2. Main result.

**Theorem 2.** Let \( k \) be any infinite cardinal. Let \( Q(k) = k \times k \), and let a partial ordering on \( Q(k) \) be defined by: \( (x_1, y_1) \preceq (x_2, y_2) \) if and only if \( x_1 \preceq x_2 \) and \( y_1 \preceq y_2 \). Then

(i) every totally unordered subset of \( Q(k) \) is finite, and

(ii) every decomposition of \( Q(k) \) is of power \( k \).

**Proof.** (i) If \( x_1 = x_2 \), then \( (x_1, y_1) \) and \( (x_2, y_2) \) are not incomparable. Hence in a totally unordered subset of \( Q(k) \) first coordinates of different members are different; they are also well ordered by \( \preceq \). Therefore, if there is an infinite totally unordered subset it would include a sequence \( (x_1, y_1), \ldots, (x_n, y_n), \ldots \) in which \( x_1 < x_2 < \cdots < x_n < x_{n+1} < \cdots \). If \( y_n \preceq y_{n+1} \) we would have \( (x_n, y_n) \preceq (x_{n+1}, y_{n+1}) \), and hence \( y_1 > y_2 > \cdots > y_n > y_{n+1} > \cdots \); but this is impossible as the \( y_n \)'s are well ordered by \( \preceq \).

(ii) Since \( |Q(k)| = k \) every decomposition of \( Q(k) \) is of power \( \leq k \). Hence it suffices to show that every decomposition is of power \( \geq k \). First assume that \( k \) is regular. For \( v < k \), let us define \( L_v = k \times \{v\} = \{(a, v): a < k\} \). If \( C \) is a chain in \( Q(k) \) such that \( C \) is cofinal in \( L_v \) and \( v < v' \), then \( C \cap L_{v'} = \emptyset \). For if \( (a, v') \in C \), then there is an \( a' \) such that \( a' \succ a \) and \( (a', v) \in C \), but \( (a, v') \) and \( (a', v) \) are incomparable. In particular, no chain is cofinal in both \( L_v \) and \( L_{v'} \) if \( v \neq v' \).

Let \( \mathcal{D} \) be any decomposition of \( Q(k) \). If for every \( v < k \) there is a \( C \in \mathcal{D} \) such that \( C \) is cofinal in \( L_v \), then it follows by the observation just made that \( |\mathcal{D}| \geq k \). If on the other hand there is a \( v \) such that no \( C \in \mathcal{D} \) is cofinal in \( L_v \), it follows from the regularity of \( k \) and from the fact that \( \bigcup \{C: C \subseteq \mathcal{D}\} \supseteq L_v \), that \( |\mathcal{D}| \geq k \).

Now if \( k \) is any infinite cardinal and \( \mathcal{D} \) is a decomposition of \( Q(k) \), then \( \{C \cap Q(k): C \in \mathcal{D}\} \) is a decomposition of \( Q(h) \) for every cardinal \( h \leq k \). Hence, for every regular cardinal \( h \) which is \( \leq k \), we have \( |\mathcal{D}| \geq h \), and therefore \( |\mathcal{D}| \geq k \). This completes the proof.

3. An application to graph theory. Let \( G \) be a set, and let \( G^2 \) denote the set of all two-element subsets of \( G \). By a graph we mean a pair \( (G, R) \), where \( G \) is a set and \( R \subseteq G^2 \). If the unordered pair \( \{x, y\} \) is an element of \( R \), we write \( xRy \); if this is not the case, we write \( x \not{R} y \).
A subset \( H \) of \( G \) is **complete** if and only if \( xRy \) for all \( x \in H, y \in H \). A subset \( H \) of \( G \) is **independent** if and only if \( x \not\sim y \) for all \( x \in H, y \in H \). A **decomposition** of a graph \( (G, R) \) is a family of mutually disjoint nonempty independent subsets of \( G \) whose union is \( G \). For any graph \( (G, R) \), we define

\[
d(G) = \text{l.u.b.}\{ | H | : H \text{ is a complete subset of } G \},
\]

\[
c(G) = \min\{ | D | : D \text{ is a decomposition of } (G, R) \}.
\]

It is clear that \( d(G) \leq c(G) \) for all graphs \( (G, R) \).

Zykov [4, Theorem 8] has shown that, given any positive integers \( d_0 \) and \( c_0 \) with \( d_0 \leq c_0 \), there is a graph \( (G, R) \) such that \( d(G) = d_0 \) and \( c(G) = c_0 \). Using Theorem 2, we now show that this result may be extended to infinite cardinals. We shall prove

**Theorem 3.** Given any infinite cardinal numbers \( k \) and \( m \) with \( k \leq m \), there exists a graph \( (G, R) \) such that \( d(G) = k \) and \( c(G) = m \).

**Proof.** Let \( P \) be any partially ordered set. If \( x, y \in P \), define \( xRy \) if and only if \( x \) and \( y \) are incomparable with respect to the partial order in \( P \). We call the graph \( (P, R) \) the **incomparability graph** of the partially ordered set \( P \).

Now, given the cardinal numbers \( k \) and \( m \), assume first that \( k = \aleph_0 \). Then the incomparability graph of the partially ordered set \( Q(m) \) satisfies the required conditions. If \( k > \aleph_0 \), we adjoin to \( Q(m) \) a set \( A \) of mutually incomparable elements with \( |A| = k \). We define \( r \leq s \) for all \( r \in A \) and \( s \in Q(m) \), and we retain the previously defined partial order within \( Q(m) \). The reader may now verify that the incomparability graph of the partially ordered set \( A \cup Q(m) \) satisfies the requirements of the theorem.

**References**


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