AN ARITHMETIC PROPERTY OF RIEMANN SUMS

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If \( f \) is a real function on the real line, periodic with period 1, define

\[
(M_n f)(x) = \frac{1}{n} \sum_{i=1}^{n} f\left(x + \frac{i}{n}\right) \quad (n = 1, 2, 3, \ldots).
\]

Writing \( \int f \) for the integral of \( f \) over \([0, 1]\), the relation

\[
\lim_{n \to \infty} (M_n f)(x) = \int f
\]

holds for all real \( x \) if \( f \) is Riemann integrable on \([0, 1]\). In the present note it is shown that there are bounded measurable functions \( f \) for which (2) is false for every \( x \) and that this convergence problem has some interesting number-theoretic aspects.

In 1934, Jessen [1] proved that if \( f \in L^1 \) on \([0, 1]\) and if \( \{n_k\} \) is an increasing sequence of positive integers in which each term divides the next, then

\[
\lim_{k \to \infty} (M_{n_k} f)(x) = \int f
\]

for almost all \( x \).

In 1948, Salem [2] showed that (3) holds for almost all \( x \) if the integral modulus of continuity of \( f \) satisfies a certain condition and if \( \{n_k\} \) satisfies a corresponding lacunarity condition. Salem's condition involves only the rate of growth of \( \{n_k\} \); no divisibility assumptions appear.

In the opposite direction, it is known (I am indebted to the referee for mentioning [4] and [5]) that there are functions \( f \in L^1 \) for which (2) fails almost everywhere. For example, if \( 0 < \alpha < 1/2 \), define

\[
f(x) = |x|^{-1+\alpha} \quad (|x| \leq 1/2)
\]

and define \( f(x) \) for all other \( x \) by periodicity. For every irrational \( x \), there are infinitely many integers \( n \) such that

\[
\left| x - \frac{m}{n} \right| < \frac{1}{n^2}
\]
for some integer $m$; if (5) holds, then $f(x - m/n) > 1/n^2 = n^{1-2\alpha}$ so that $(M_nf)(x) > n^{1-2\alpha}$. Thus

$$\limsup_{n \to \infty} (M_nf)(x) = +\infty$$

for almost all $x$. If $p < 2$, we may choose $\alpha$ so that $1 - 1/p < \alpha < 1/2$, and thus get examples of $f \in L^p$ for which (6) holds almost everywhere.

This crude method does not settle the problem for $L^2$, nor, a fortiori, for bounded measurable functions. However, (2) fails even there, and it turns out that arithmetic properties of $\{n_k\}$ are crucial; Remark (A) at the end of this note makes this very evident.

**Theorem.** Let $S$ be a sequence of positive integers which contains sets $S_N$ ($N = 1, 2, 3, \ldots$), each consisting of $N$ terms, such that no member of $S_N$ divides the least common multiple of the other members of $S_N$.

Then to every $\epsilon > 0$ there exists a bounded measurable function $f$, periodic with period 1, such that $0 \leq f \leq 1$, and such that

$$\limsup_{n \to \infty} (M_nf)(x) \geq \frac{1}{2}$$

for all $x$, although $\int f \leq \epsilon$.

For instance, $S$ could be any sequence of primes.

In the proof, $f$ is constructed as the characteristic function of an open set.

**Proof.** We may assume, without loss of generality, that the sets $S_N$ are pairwise disjoint.

Fix $N > 2$, choose $\delta > 0$ such that $\delta^N = N^{-1}(\log N)^{-2}$, let $g$ and $h$ be the characteristic functions of sets $G$ and $H$, where $G$ is the union of the segments $(k, k + \delta)$ ($k = 0, \pm 1, \pm 2, \ldots$), and $H$ is the complement of $G$. If $n_1, \ldots, n_N$ are the members of $S_N$, let $k_i$ be the least common multiple of $n_i, \ldots, n_{i-1}, n_{i+1}, \ldots, n_N$, for $i = 1, \ldots, N$. We can then find integers $p_1, \ldots, p_N$, each so much larger than the preceding one that the following is true: if

$$\phi_N(l) = \prod_{i=1}^{N} g(k_i p_i),$$

$$\psi_{j,N}(l) = h(k_j p_i) \prod_{i \neq j} g(k_i p_i) \quad (1 \leq j \leq N),$$

then $\int \phi_N$ and $\int \psi_{j,N}$ differ by as little as we please from the products

$$\left( \int g \right)^N = \delta^N$$

and
Put $A_N = B_{1,N} \cup \cdots \cup B_{N,N}$, where $B_{j,N}$ is the set whose characteristic function is $\psi_{j,N}$. Since the sets $B_{1,N}, \cdots, B_{N,N}$ are pairwise disjoint, and since

$$N(1 - \delta)^{N-1} = N\delta^N \left(\frac{1}{\delta} - 1\right) > N\delta^N(N^{1/N} - 1) > \delta^N \log N,$$

we see from (7) and (8) that $p_1, \cdots, p_N$ can be so chosen that

$$\sum_{n=1}^{N} m(A_n) > \frac{1}{N \log N},$$

where $m(A_N)$ denotes the Lebesgue measure of $A_N \cap [0, 1]$. Moreover, $p_1, \cdots, p_N$ can be chosen to be primes which divide no member of $S_N$

Suppose now that $x \in B_{j,N}$. Then $k_ip_jx \in G$ if $i \neq j$, and $k_jp_jx \in H$. Since $n_j$ divides $k_i$ if $i \neq j$, we have

$$k_i p_i \left( x + \frac{r}{n_j} \right) \equiv k_i p_i x \pmod{1}$$

if $i \neq j$ and $r = 1, \cdots, n_j$. But $n_j$ does not divide $k_j p_j$ (this is where the arithmetic hypothesis imposed on $S_N$ is used), and therefore the terms of the arithmetic progression

$$k_j p_j \left( x + \frac{r}{n_j} \right) \pmod{1}$$

for all $x \in B_{j,N}$, and (12) implies that

$$\max_{n \in S_N} (M_n \phi_N)(x) \geq \frac{1}{2} \quad (x \in A_N).$$

By (9), $\sum m(A_N) = \infty$. Hence there is a sequence $\{\alpha_N\}$ of real numbers such that almost every $x$ lies in infinitely many of the translates $A_N + \alpha_N$ (see [3, p. 165, Lemma 1.24]). Choose $N_0$ so that

$$4 \sum_{N_0}^{\infty} N^{-1}(\log N)^{-2} < \epsilon$$

and put
\[
\phi(t) = \sup_{N \geq N_0} \phi_N(t - \alpha_N).
\]

Then (13) implies
\[
\limsup_{n \in S} (M_n \phi)(x) \geq \frac{1}{2}
\]
for almost all \(x\), although \(\int \phi \leq \epsilon/2\), by (9) and (14).

If now \(E\) is the set of measure 0 on which (16) fails, let \(\chi\) be the characteristic function of a periodic open set \(V\) which contains \(E + r\) for all rational numbers \(r\), and such that \(m(V) < \epsilon/2\). Setting \(f = \max(\phi, \chi)\), we obtain a function which has the properties asserted by the theorem.

**Remarks.** (A) There are sequences \(\{n_k\}\) which satisfy the hypothesis of Jessen's theorem but such that \(\{1 + n_k\}\) is a sequence of primes. To see this, suppose \(n_k\) is chosen; by Dirichlet's theorem on primes in arithmetic progressions, there is an integer \(r > 1\) such that \(q = 1 + rn_k\) is prime; put \(n_{k+1} = rn_k\).

Thus \(\{(M_{n_k} f)(x)\}\) converges to \(f\) a.e. although the sequence \(\{(M_{1+n_k} f)(x)\}\) need not do so.

(B) Take \(\epsilon_k = 2^{-k}\) in our theorem, let \(f_k (f = 1, 2, 3, \cdots)\) be the corresponding functions, and put \(F = \sum k f_k\). Then \(F \in L^p\) on \([0, 1]\) for every \(p < \infty\), but
\[
\limsup_{n \in S} (M_n F)(x) = +\infty
\]
for every \(x\).

(C) It is easy to see that \(M_nf \rightarrow f\) in the \(L^p\)-norm, as \(n \rightarrow \infty\), for every \(f \in L^p\), if \(1 \leq p < \infty\). Hence for every \(f \in L^1\) there is a sequence \(\{n_k\}\) such that (3) holds almost everywhere.

**References**


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