AN ISOPERIMETRIC INEQUALITY

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If \( f \) maps the \( k \)-cube \( I^k \) into \( R^n \) so that the images of opposite \( k-1 \) faces of \( I^k \) are at least distance \( b \) apart, it is plausible that the volume of \( f \) can be no smaller than \( b^k \). This is true if \( f \) is Lipschitzian and is established as a corollary to the more general theorem below. This problem was suggested by D. C. Spencer, who together with S. Bergman proved it for \( k = 2 \) in [2].

DEFINITIONS.

1. \( I_*(R^n) = \bigoplus I_j(R^n) \) is the chain complex of integral currents in \( R^n \) with boundary homomorphism \( \partial \) as defined in [3]. The mass of a current \( T \) is written \( M(T) \). For \( T \subseteq I_0(R^n) \) let \( M_0(T) \) be the absolute value of the coefficient sum of \( T \).

2. \( \sigma^k \) is the cell complex of the \( k \)-cube \( I^k \), i.e., the chain complex generated by the cubical faces of \( I^k \) of various dimensions. \( \sigma^1 \) is generated by \{ \([0, 1], [0], [1] \) \} with \( d[0, 1] = [1] - [0] \), \( d[0] = d[1] = 0 \). \( \sigma^2 \) is generated by \{ \([0, 1] \times [0, 1], [0, 1] \times [1], [0] \times [0], \) \dots \}, etc. Let \( \alpha^k = [0, 1] \times \cdots \times [0, 1] \) be the unique \( k \) cell in \( \sigma^k \), and let \( \alpha^k(i, \epsilon), i = 1, 2, \cdots, k, \epsilon = 0, 1 \) denote the \( k-1 \) cell obtained by setting the \( i \)th coordinate equal to \( \epsilon \).

3. For \( A \subseteq R^n \) define \( u_A : R^n \to R^n, u_A(x) = \text{distance} (x, A) \). Set \( U_r = R^n \cap \{ x : u_A(x) < r \} \) for \( r \in R \). Note that \( u_A \) satisfies a Lipschitz condition with constant 1 for any \( A \).

THEOREM. Let \( F : \sigma^k \to I_*(R^n) \) be a chain map of degree 0 such that

1. For some 0 cell (vertex) \( v \in \sigma^k \), \( M_0(F(v)) \geq 1 \).
2. \( \text{inf} \{ |x - y| : x \in \text{support} (F(\beta)), y \in \text{support} (F(\gamma)) \} \geq b \), whenever \( \beta \) is a face of \( \alpha^k(i, 0) \) and \( \gamma \) is a face of \( \alpha^k(i, 1) \) for some \( i = 1, 2, \cdots, k \).

Then \( M(F(\alpha^k)) \geq M_0(F(v)) \cdot \prod_{i=1}^k b_i \).

LEMMA. Let \( A \) and \( B \) be subsets of \( R^n \) with \( \text{inf} \{ |x - y| : x \in A, y \in B \} = b \). Suppose \( S \subseteq I_0(R^n) \), \( \text{support} (S) \subseteq A \), \( \text{support} (T) \subseteq B \), and \( M_0(S) \geq 1 \). Suppose also \( Q \subseteq I_1(R^n) \) with \( \partial Q = S - T \). Then \( M_0(S) = M_0(T) \) and \( M(Q) \geq M_0(S) \cdot b \). Also, for \( L \) almost all \( r \in (0, b) \), \( \partial (Q \cap U_r(A)) = \partial Q \cap U_r(A) \in I_0(R^n) \) and

\[ M_0(\partial (Q \cap U_r(A))) - \partial Q \cap U_r(A) \leq M_0(S). \]

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Proof. The lemma follows in part from [3, 3.8(3), 3.9, 3.10, 8.14] and the rest is obvious.

Proof of the theorem. By induction. For $k = 1$, the theorem is implied by the lemma. Assume the theorem holds up to dimension $k - 1$. Consider the map

$$G_r: I^{k-1} \rightarrow I_*(R^n)$$

$$G_r(\beta) = \partial[F([0, 1] \otimes \beta) \cap U_r(\text{support}[F(\alpha^k(1, 0))])]$$

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for each cell $\beta \subseteq \partial^{k-1}$. By [3; 3.8(3), 3.9, 3.10, 8.14], $G_r$ is defined for $L_1$ almost all $r \in (0, b_1)$. One verifies that for each such $r$, $G_r$ is a chain map. Using the lemma, it follows that each such $G_r$ satisfies the induction hypothesis. Thus

$$M(G_r(\alpha^{k-1})) \geq M_0(F(v)) \cdot \prod_{i=2}^{k} b_i.$$

Using [3, 3.10] one sees

$$M(F(\alpha^k)) \leq \int_0^{b_1} M(G_r(\alpha^{k-1}))dr \leq M_0(F(v)) \cdot \prod_{i=1}^{k} b_i.$$

Corollary. Let $f: I^k \rightarrow R^n$ be Lipschitzian with $|f(x) - f(y)| \leq b$ whenever $x$ and $y$ lie on opposite $k-1$ dimensional faces of $I^k$. The restrictions of $f$ to the various faces of $I^k$ determine integral currents in $R^n$, and since volume is always greater than or equal to mass, one has

$$\text{volume}(f) \geq M(f_*(I^k)) \leq b^k.$$

References

1. F. J. Almgren, Jr., The homotopy groups of the integral cycle groups, Topology 1 (1962), 257–299.

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