THE COEFFICIENTS OF THE RECIPROCAL OF A BESSEL FUNCTION

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Put

\[ \left\{ \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n!} \right\}^{-1} = \sum_{n=0}^{\infty} \frac{\omega_n x^n}{n!} \]

This is equivalent to

\[ \sum_{r=0}^{n} (-1)^r \binom{n}{r} \omega_r = \begin{cases} 1 & (n = 0) \\ 0 & (n > 0) \end{cases} \]

In a letter to the author, J. Riordan has raised the question whether the \( \omega_n \) can satisfy a recurrence of order independent of \( n \). We shall show that the \( \omega_n \) cannot satisfy a recurrence order \( k \), where \( k \) is independent of \( n \), with polynomial coefficients. More precisely we show that the assumption

\[ \sum_{j=0}^{k} A_j(n) \omega_{n+j} = 0 \quad (n > N), \]

where the \( A_j(n) \) are polynomials in \( n \) with complex coefficients and \( k, N \) are fixed, leads to a contradiction.

Since it is no more difficult, we consider the following more general problem. Put

\[ \left\{ \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n! \Gamma(n + 1)} \right\}^{-1} = \sum_{n=0}^{\infty} \frac{\omega_n(v) x^n}{n! \Gamma(n + 1)} \]

This is equivalent to

\[ \sum_{r=0}^{n} (-1)^r \binom{n}{r} \binom{n}{n+r} \omega_r(v) = \begin{cases} 1 & (n = 0) \\ 0 & (n > 0) \end{cases} \]

We assume that \( v \) is not a negative integer; then it is clear that the \( \omega_n(v) \) are uniquely determined by (3) or (4).

Now assume that the \( \omega_n(v) \) satisfy the recurrence

\[ \sum_{j=0}^{k} A_j(n, v) \omega_{n+j}(v) = 0 \]
for all \( n > N \), where the \( A_j(n, \nu) \) are polynomials in \( n \) with complex coefficients and \( k, N \) are fixed. Put

\[
f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n! \Gamma(\nu + n + 1)},
\]

\[
g(x) = \frac{1}{f(x)} = \sum_{n=0}^{\infty} \frac{\omega_n(\nu) x^n}{n! \Gamma(\nu + n + 1)}.
\]

Now if \( P(x) \) is an arbitrary polynomial with constant coefficients, it is evident that

\[
P(xD)g(x) = \sum_{n=0}^{\infty} P(n) \frac{\omega_n(\nu) x^n}{n! \Gamma(\nu + n + 1)},
\]

where \( D = d/dx \); moreover since

\[
Dg(x) = \sum_{n=0}^{\infty} \frac{\omega_{n+1}(\nu) x^n}{n! \Gamma(\nu + n + 1)},
\]

it follows that

\[
(6) \quad P(xD) \cdot Dg(x) = \sum_{n=0}^{\infty} \frac{P(n)}{(\nu + n + 1)_j} \frac{\omega_{n+1}(\nu) x^n}{n! \Gamma(\nu + n + 1)}.
\]

If we multiply both sides of (5) by

\[
\frac{x^n}{n! \Gamma(\nu + n + 1)}
\]

and sum over all \( n > N \) we get

\[
(7) \quad \sum_{j=0}^{k} \sum_{n=0}^{\infty} A_j(n, \nu) \frac{\omega_{n+1}(\nu) x^n}{n! \Gamma(\nu + n + 1)} = C(x),
\]

where \( C(x) \) is a polynomial in \( x \) of degree \( \leq N \). Repeated differentiation of (7) leads to an equation of the same kind in which the right member vanishes.

Comparison of (7) with (6) shows that \( g(x) \) satisfies a differential equation of the form

\[
(8) \quad \sum_{j=0}^{m} B_j(x, \nu) D^{m-i}g(x) = 0,
\]

where the \( B_j(x, \nu) \) are polynomials in \( x \). The order \( m \) depends upon the degree of the \( A_j(n, \nu) \). We may assume that

\[
(9) \quad B_0(x, \nu) \neq 0.
\]
In the next place since \( g(x) = \frac{1}{f(x)} \), we have

\[
g'(x) = -\frac{f'(x)}{f^2(x)}, \quad g''(x) = -\frac{f''(x)}{f^3(x)} + \frac{2(f'(x))^2}{f^4(x)},
\]

\[
g'''(x) = -\frac{f'''(x)}{f^4(x)} + \frac{6f'(x)f''(x)}{f^5(x)} - \frac{6(f'(x))^3}{f^6(x)},
\]

and so on. Making use of (10) we may replace (8) by a differential equation in \( f(x) \).

For simplicity we shall assume \( m = 3 \); the method is however quite general. We find that

\[
B_0\left\{ -f^2(x)f'''(x) + 6f(x)f'(x)f''(x) - 6(f'(x))^3 \right\} + B_1\left\{ -f^2(x)f''(x) + 2f(x)(f'(x))^2 \right\} - B_2f^3(x)f'(x) + B_3f^3(x) = 0,
\]

where \( B_j = B_j(x, \nu) \). Now, on the other hand, we have

\[
xf''(x) + (\nu + 1)f'(x) + f(x) = 0,
\]

so that

\[
xf'''(x) + (\nu + 2)f''(x) + f'(x) = 0.
\]

We may eliminate \( f''(x) \) and \( f'''(x) \) in (11); there results an equation of the form

\[
C_0(x, \nu)(f'(x))^3 + C_1(x, \nu)(f'(x))^2f(x)
\]

\[
+ C_2(x, \nu)f'(x)f^2(x) + C_3(x, \nu)f^3(x) = 0,
\]

where \( C_j(x, \nu) \) are polynomials in \( x \). Moreover, by (9) \( C_6(x, \nu) = -6B_0(x, \nu) \neq 0 \).

It therefore follows from (12) that \( f'(x)/f(x) \) is an algebraic function of \( x \). However, since \( f(x) \) has infinitely many zeros, it follows that the logarithmic derivative \( f'(x)/f(x) \) has infinitely many poles and therefore cannot be an algebraic function.

We have proved the following

**Theorem.** Let \( \nu \) be an arbitrary complex number not equal to a negative integer and define \( \omega_n(\nu) \) by means of (3). Then \( \omega_n(\nu) \) cannot satisfy a recurrence

\[
\sum_{j=0}^{k} A_j(n, \nu)\omega_{n+j}(x) = 0 \quad (n > N),
\]

where the \( A_j(n, \nu) \) are polynomials in \( n \) with complex coefficients and \( k, N \) are fixed.

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