ON BOUNDED FUNCTIONS WITH ALMOST-PERIODIC DIFFERENCES

F. W. CARROLL

Let $G$ be a multiplicative group, and let $f$ be a complex-valued function on $G$. The left differences of $f$ are the functions $\Delta_x f$, $\Delta_x f(x) = f(hx) - f(x)$, $(h, x \in G)$. The following proposition is known [1; 2, p. 286], and is easy to prove.

**Lemma 1.** If $G_1$ is any topological group, if $f_1$ is a complex-valued function on $G_1$, and if all the left differences of $f_1$ are continuous, then either $f_1$ is continuous or else $f_1$ is unbounded on every nonempty open subset of $G_1$.

One of several equivalent definitions of right almost-periodicity is the following:

**Definition.** A complex-valued function $f$ on a group $G$ is right almost-periodic if, to every positive $\varepsilon$, there corresponds a finite number of elements of $G$, say $s_1, \ldots, s_k$, such that to every $x$ in $G$ we can associate an integer $i \leq k$ for which

$$|f(xt) - f(xs_i)| < \varepsilon,$$

for all $x \in G$.

Doss proved the following theorem [3]:

**Theorem.** Let $G$ be a multiplicative group, and let the left differences $\Delta_x f$ be right almost-periodic for every $h \in G$, where $f$ is a given complex-valued function on $G$. If $f$ is bounded, then $f$ is a right almost-periodic function.

The proof given by Doss was elementary, depending only on the definition of right almost-periodicity. On the other hand, the similarity of the statements of Lemma 1 and the theorem suggests that the latter may follow naturally from the former. This is the case. The crucial tool is the following well-known result [4, p. 168].

**Lemma 2.** If $G$ is any topological group, there is a compact group $M$ and a continuous homomorphism $\alpha$ of $G$ onto a dense subgroup $G_1$ of $M$ such that a (continuous) function $f$ on $G$ is right almost-periodic if and only if there is a continuous function $g$ on $M$ such that $f(x) = g(\alpha(x))$ for all $x$ in $G$.

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Proof of the theorem. Consider the given group $G$ as a discrete topological group, and let $M$, $\alpha$, and $G_1$ be as in Lemma 2. Let $H$ denote the kernel of $\alpha$. For each $h \in G$, $\Delta_h f$ is right almost-periodic on $G$, and can therefore be considered as a function on $G_1$. That is

$$\Delta_h f(kx) = \Delta_h f(x), \quad \text{for all } k \in H, x, h \in G.$$ 

Taking $x = 1$, we obtain

$$f(hk) - f(h) = f(k) - f(1) \quad (k \in H, h \in G). \quad (1)$$

Taking $h = 1, k, \ldots, k^{n-1}$ in (1) and adding the left and right-hand sides, respectively, of the resulting equations, we obtain

$$f(k^n) - f(1) = n[f(k) - f(1)] \quad (k \in H). \quad (2)$$

Since $f$ is bounded, it follows that $f(k) - f(1) = 0$. Thus (1) shows that a function $f_1$ can be defined on $G_1$ such that $f_1(\alpha(x)) = f(x)$ for all $x$ in $G$. In view of Lemma 2, it suffices to prove that $f_1$ has an extension as a continuous function on all of $M$.

Trivially, the left differences of $f_1$ correspond to left differences of $f$ so that, by Lemma 2, the left differences of $f_1$ have continuous extensions to all of $M$. In particular, they are uniformly continuous on $G_1$. By Lemma 1, then, $f_1$ is continuous on $G_1$. Now let $y \in M$ and $\epsilon > 0$ be given. There is an open neighborhood $V$ of 1 in $M$ such that

$$|f_1(y_2) - f_1(y_1)| < \epsilon, \quad \text{for all } y_1, y_2 \in V \cap G_1. \quad (3)$$

Let $s$ be an element of $G_1$ such that $y \in sV$. Since $\Delta_s f_1$ is uniformly continuous on $G_1$, there is a symmetric neighborhood $W$ of 1 in $M$ such that $yW \subset sV$ and

$$|\Delta_s f_1(z_2) - \Delta_s f_1(z_1)| < \epsilon \quad (z_1, z_2 \in G_1, s^{-1}z_2 \in W^2). \quad (4)$$

Let $t_1, t_2$ be any two points of $yW \cap G_1$. Then we have

$$|f_1(t_2) - f_1(t_1)| = |f_1(ss^{-1}t_2) - f_1(ss^{-1}t_1)| = |\Delta_s f_1(s^{-1}t_2) - \Delta_s f_1(s^{-1}t_1) + f_1(s^{-1}t_2) - f_1(s^{-1}t_1)| < 2\epsilon,$$

by (3) and (4). Hence $f_1$ has a continuous extension to all of $M$, and the theorem follows.

References

1. N. G. de Bruijn, Functions whose differences belong to a given class, Nieuw Arch. Wisk. (2) 23 (1951), 194–218.
RESTRICTIONS OF FOURIER-STIELTJES TRANSFORMS

JAMES WELLS

1. Let $G$ be a locally compact group with dual group $\Gamma$, and let $dx$ and $dy$ be the Haar measures on $G$ and $\Gamma$, respectively. For a function $f \in L^1(G)$, the group algebra of $G$, the Fourier transform of $f$ is denoted by $\hat{f}$:

$$\hat{f}(y) = \int_G (x, y) f(x) dx;$$

and for a measure $\mu \in M(G)$, the algebra of bounded measures on $G$, the Fourier-Stieltjes transform of $\mu$ is denoted by $\hat{\mu}$:

$$\hat{\mu}(y) = \int_G (x, y) d\mu(x).$$

Here $(x, y)$ denotes the value of the character $y \in \Gamma$ at the point $x \in G$. Let $A$ denote the family of Fourier transforms of functions $f \in L^1(G)$. For $F \subseteq \Gamma$, $\hat{f} \mid F$ denotes the restriction of $\hat{f}$ to $F$ and $A \mid F = \{\hat{f} \mid F : \hat{f} \in A\}$. A function $\varphi$ on $\Gamma$ is said to be a multiplier of $A$ provided $\varphi A \subseteq A$. It is a theorem of Helson's [2, Theorem 1] that the multipliers of $A$ are precisely the Fourier-Stieltjes transforms. We are going to show that the obvious analogue persists on closed subsets of $\Gamma$, i.e., the multipliers of $A \mid F$ are precisely the almost everywhere restrictions to $F$ of Fourier-Stieltjes transforms.

**Theorem.** Suppose $\varphi$ is a function on $\Gamma$ and $F \subseteq \Gamma$ is closed. In order that $\varphi \mid F = \hat{\mu} \mid F$ almost everywhere for some $\mu \in M(G)$, it is necessary and sufficient that

$$(H) \quad \varphi A \mid F \subseteq A \mid F.$$

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