GÖDEL NUMBERINGS VERSUS FRIEDBERG NUMBERINGS

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In [3], Rogers discussed the concept of Gödel numbering. He defined a semi-effective numbering, constructed a semi-lattice of equivalence classes of semi-effective numberings, and showed that all Gödel numberings belong to the unique maximal element of this semi-lattice.

In [1], Friedberg gave a recursive enumeration without repetition of the set of partial recursive functions of a single variable. Friedberg's numbering is clearly a semi-effective numbering which is not a Gödel numbering.2

QUESTION. How do Friedberg numberings (Definition 1 below) compare with Gödel numberings? More generally, where do Friedberg numberings fit into Rogers' semi-lattice?

Definition 1. A Friedberg numbering $\tau$ is a semi-effective numbering such that

1. $D_\tau = N$,
2. $\tau_i \neq \tau_j$ if $i \neq j$.

Summary of results.3

I. If $\tau$ is a Friedberg numbering then $[\tau]$ is a minimal element of Rogers' semi-lattice. (Theorem 1.)

II. There exists two Friedberg numberings $\tau_1$ and $\tau_2$ such that $[\tau_1]$ and $[\tau_2]$ are incomparable. (Theorem 2.)

As a consequence of I and II we see that Rogers' semi-lattice is not a lattice (which answers a question raised in [3]).

Theorem 1. Let $\rho$ be a semi-effective numbering, $\tau$ a Friedberg numbering. If $\rho \subseteq \tau$, then $[\rho] = [\tau]$.

Proof. Since $\rho \subseteq \tau$, there is a recursive function $g$, mapping $D_\rho$ onto $D_\tau$ (which equals $N$) such that $\rho_x = \tau_{g(x)}$ on $D_\tau$. Define $h$ by

$$h(i) = \mu y [g(y) = i \cdot y \in D_\tau]$$

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2 For example Friedberg's $\tau$ does not have the property that given any recursively enumerable sequence $f_1, f_2, \ldots$ of partial recursive functions there is a general recursive function $h$ such that $\tau_{h(n)} = f_n$. Furthermore, $\tau$ does not satisfy the recursion theorem relative to $\tau$. For there exists a recursive function $h$ such that for all $n$, it is not the case that $\tau_{h(n)} = \tau_n$ (e.g. let $h(n) = 1 - \text{sgn}(n)$).

3 We follow the notation of [3].

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Theorem 2. There exists two Friedberg numberings \( \rho \) and \( \tau \) such that \([\rho] \neq [\tau]\).

**Proof.** Let \( \rho \) be the Friedberg numbering constructed by Friedberg [1, corollary to Theorem 3] from the standard Gödel numbering \( \pi \). Our construction of \( \tau \) from \( \rho \) is similar to Friedberg's construction of \( \rho \) from \( \pi \). However Case 1 of the priority scheme is essentially different.

Since \( \rho \) is a semi-effective numbering and \( \pi \) is the standard numbering, there is a recursive function \( h \) such that

\[
\rho_i = \pi_{h(i)}.
\]

As in [1] 0 will always be unused and \( \rho_0 = \tau_0 = \mathbf{u} \). Let \( e_a \) be the value of \( e \) which is being pursued at step \( a \). Then the following three cases arise.

**Case 1.** \((\exists y)[y < a \cdot T(h(e_a), e_a, y) \cdot U(y) = x]. x \) is a follower of \( e_a \).\(^5\)

Then release \( x \).

**Case 2.** Case 1 does not hold and there exists an \( x \) such that \( \rho^a_x \) is identical with \( \tau^a_x \). If this \( x \) is free, then either \( x \leq e_a \) or \( x \) has been previously displaced by \( e_a \). If this \( x \) is a follower, it is a follower of some \( e \leq e_a \). Otherwise \( x = 0 \).

Then do nothing.

**Case 3.** Neither Case 1 nor Case 2 holds.

Then the following four acts, some of which may be vacuous, are carried out.

A. If \( e_a \) has no follower, make the lowest unused \( x \) other than 0 a follower of \( e_a \).

B. Put into \( \tau_x \) where \( x \) is the follower of \( e_a \) all the members of \( \rho^a_x \). (Thus \( \tau_x \) becomes identical to \( \rho^a_x \).)

C. If for some \( x' \neq x \), \( \tau^a_{x'} \) is identical with \( \rho^a_{e_a} \), then put into \( \tau^a_{x'} \) the smallest ordered pair (in some standard enumeration of ordered pairs) which differs in its argument member from any ordered pair previously listed in any \( \rho \) or \( \tau \). Thus \( x' \) is displaced by \( e_a \).

D. If the \( x' \) of act C is a follower of some \( e' \), release it.

The proof proceeds by means of four lemmas. The first three, modifications of Lemmas 3, 4, 5 of [1], respectively, show that \( \tau \) is a Friedberg numbering. Lemma 4 shows that \( \rho \) and \( \tau \) are incomparable Friedberg numberings. For the sake of completeness we give all details. (Note that our Lemmas 1 and 3 are actually simpler than the corresponding ones of [1].)

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\(^4\) \( \mathbf{u} \) is the everywhere undefined function.  
\(^5\) \( T \) and \( U \) are defined in [2].
**Lemma 1.** For every \( e \) there is an \( x \) such that \( \tau_0 = e^{-1} \).

**Proof.** We first show that \( e \) cannot have an infinite number of disloyal followers. For \( e \) can lose a follower only by Case 1 or Case 3. Clearly \( e \) can lose a follower at most once by Case 1. Furthermore, since \( p \) is a Friedberg numbering, there exists a stage \( a \) such that for every \( e' < e \), either \( p_{e'} \) has acquired an ordered pair never to be acquired by \( p_e \) or \( p_{e'} \) has acquired an ordered pair never to be acquired by \( p_e \). In the first case, \( \tau_{x-1}^e \) can never be equal to \( p_{e'} \) when \( x \) is a follower of \( e \) and \( a' \geq a \). In the second case, if \( e \) should acquire a new follower \( x \), \( \tau_0 \) will acquire all members in \( p_e \) and \( \tau_{x-1}^e \) cannot ever again be identical with \( p_{e'} \). Thus \( e \) cannot have infinitely many disloyal followers.

Let \( a_0 \) be a step after which \( e \) will never lose a follower. If Case 3 occurs infinitely often with \( e_0 = e \), then the first time it occurs after step \( a_0 \), \( e \) either acquires a follower or it already has one. This follower (call it \( x \)) will be loyal. Hence \( \tau_0 = e^{-1} \).

Suppose Case 3 occurs only finitely often with \( e_0 = e \). Then since Case 1 can occur at most once, Case 2 must occur infinitely often with \( e_0 = e \). Whenever Case 2 occurs there must be an \( x \) such that \( \tau_{x-1}^e = p_x \). Furthermore this \( x \) is either a follower of some \( e' \leq e \) or it must be free or it must be 0. (Recall that if Case 2 occurs when \( x \) is free, either \( x \leq e \) or \( x \) has previously been displaced by \( e \).) If \( x = 0 \), \( \tau_0 = e_0 \) is free. So assume \( x > 0 \) and \( x \) is either free as required by Case 2 or a follower of some \( e' \leq e \). Our aim is to show that \( x \) can take only a finite number of different values if \( e = e_0 \) and Case 2 occurs infinitely often.

Only a finite number of values of \( x \) are \( \leq e \). By hypothesis, Case 3 occurs only finitely often with \( e = e_0 \). Hence only finitely many \( x \)'s are displaced by \( e \). Hence there are only finitely many \( x \)'s such that \( x \) is free.

Suppose \( x \) is a follower of \( e' \leq e \). There are only finitely many followers of \( e \) for each \( e \) by the construction above. Hence there are only finitely many followers of \( e' \leq e \).

Thus only finitely many different numbers can serve as the \( x \) of Case 2 with \( e = e_0 \). Since Case 2 arises infinitely often, there must be an \( x \) such that \( \tau_{x-1}^e = p_{e'} \) infinitely often. Thus \( \tau_0 = e' \).

Thus we see that 0 is the only value of \( x \) which remains unused.

**Lemma 2.** If \( x \neq x' \), \( \tau_0 \) and \( \tau_0 \) are not the same finite function.

**Proof.** By construction (see Case 3).

**Lemma 3.** If \( x \neq x' \) and \( \tau_0 \) and \( \tau_0 \) have infinite domains, then \( \tau_0 \) and \( \tau_0 \) are not the same function.
Proof. If $x > 0$, $x$ must become a follower of some $e$. If $x$ is a disloyal follower, then after $x$ is released, $x$ can only acquire a new member when $x$ is displaced. But $x$ is never displaced by any $e \geq x$ and only once by an $e < x$. Hence if $x$ is disloyal, $r_x$ has a finite domain. Thus if $r_x$ has an infinite domain, $x$ must be a loyal follower of some $e$. Similarly $x'$ must be a loyal follower of some $e'$. Thus $r_x = p_x$ and $r_{x'} = p_{x'}$. Since $x \neq x'$, $e \neq e'$. Furthermore, since $p$ is a Friedberg numbering $p_x \neq p_{x'}$ if $e \neq e'$. Thus $r_x \neq r_{x'}$.

Lemma 4. $\tau$ and $p$ are incomparable Friedberg numberings.

Proof. Suppose $\tau$ and $p$ are comparable. Then since $\tau$ and $p$ are Friedberg numberings there would exist a recursive permutation $g$ such that $p_i = \tau_g(i)$ for all $i$. Now $g = p_\tilde{x}$ for some $\tilde{x}$, and $r_{\tilde{x}} = \tau_{g(\tilde{x})} = p_\tilde{x}$ for $\tilde{x} = g(\tilde{e})$. Since $g$ is a recursive permutation, it has an infinite domain. Hence $\tilde{x}$ is a loyal follower of $\tilde{e}$. (See the proof of Lemma 3.)

Now let $a$ be the first step such that
(a) $e_a = \tilde{e}$,
(b) $\hat{x}$ is a follower of $\tilde{e}$,
(c) $(\exists y)(y < a \cdot T(h(\tilde{e}), \tilde{e}, y) \cdot U(y) = \hat{x})$.

Such a step must exist since $p_{\tilde{e}(0)} = p_{\tilde{e}} = \tau_{\tilde{e}}$ and hence $\pi_{\tilde{e}(0)}(\tilde{e}) = p_{\tilde{e}}(\tilde{e}) = \tau_{\tilde{e}}(\tilde{e}) = \hat{x}$. Thus there is a $y$ such that $T(h(\tilde{e}), \tilde{e}, y)$ and $U(y) = \hat{x}$. Thus Case 1 would be in order and $\hat{x}$ would be released—contradiction.

Remark 1. In Lemma 4 we prove something stronger than mere incomparability—there is no partial recursive function $g$ such that $\rho_i = \tau_{\tilde{e}(i)}$ whenever $g$ is defined and $g$ is defined for all $i$ such that $p_i$ has an infinite domain.

Corollary 1. Rogers’ semi-lattice is not a lattice.

Remark 2. Let $P = [\rho] \cup [\tau]$. Then $P$ contains a semi-effective numbering in which each partial recursive function is repeated exactly twice. However, $P$ does not contain a Friedberg numbering.

Remark 3. It is easy to see that for every $k$, $\rho$ contains a numbering $\rho^*$ such that $D_\rho = N$ and each partial recursive function is repeated exactly $k$ times. (More generally, it is easy to show that $[\rho]$ contains a numbering in which each partial recursive function is repeated infinitely many times.) Hence there cannot be a 1-1 mapping $g$ of $D_\rho$ onto $D_\rho$ such that $\rho_i = p_{0(0)}$ even though $[\rho^*] = [\rho]$. This observation answers another question raised by Rogers [3, p. 336].

Open Problem. Let $Q$ be a minimal element of Rogers’ semi-lattice. Does there exist a Friedberg numbering $\tau$ such that $\tau \in Q$?

In conclusion it may be remarked that both Theorem 2 and the
fact that a Friedberg numbering $\tau$ does not satisfy the recursion theorem relative to $\tau$ (footnote 2) may provide some limitation on the usefulness of replacing Gödel numberings by Friedberg numberings in certain technical investigations.

**Bibliography**


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**THE CLOSED CONVEX HULL OF CERTAIN EXTREME POINTS**

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Let $H$ be a Hausdorff space and let $C(H)$ be the space of bounded continuous real-valued functions on $H$, $C(H)$ having the usual supremum norm. Also, let $S$ be the unit ball of $C(H)$, let $E$ be the set of extreme points of $S$, and let $K$ be the set of characteristic functions of open-and-closed sets in $H$.

It is known [1, §3, p. 10] that when $H$ is compact, $S$ is the norm closed convex hull of $E$ if and only if $H$ is totally disconnected. It is known also [4, p. 103] that if $H$ is extremally disconnected, then $S$ is the norm closed convex hull of $E$. The purpose of this paper is to give a necessary and sufficient condition for $S$ to be the norm closed convex hull of $E$. The theorem for compact $H$ mentioned above is a special case of our result (cf. [3, p. 247]). We are indebted to the referee for observing that if $H$ is completely regular, our result follows at once from the compact case (cf. [3, p. 88]). Our condition is also necessary and sufficient for each of the subsets $E$ and $K$ to be fundamental (cf. [2, p. 35]) in $C(H)$.

If $f$ is a function in $C(H)$, we shall call the set

$$Z(f) = \{ h : h \in H, f(h) = 0 \}$$

the zero-set of the function $f$. We observe that if $r$ is a real number, then the sets $\{ h : f(h) = r \}$, $\{ h : f(h) \geq r \}$, and $\{ h : f(h) \leq r \}$ are zero-sets.

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