ON THE STRUCTURE OF THE GREEN'S OPERATOR

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1. Introduction. In the study of Cauchy problems of the form

\begin{equation}
\frac{du}{dt} + Au = f; \quad u(t) = T
\end{equation}

(where for example: \( t \rightarrow u(t) \in \mathcal{C}(H) \) on \( [\tau, b] \); \( t \rightarrow u(t) \in \mathcal{C}^k(D(A)) \) on \( [\tau, b] \); \( H \) is a Hilbert space; \( -A \) is a closed (unbounded) operator, infinitesimal generator of a strongly continuous semi-group; \( \mathcal{C}^k(H) \) is the space of \( k \)-times continuously differentiable functions of \( t \) with values in \( H \); the domain of \( A \), \( D(A) \), has the graph topology; and \( f, T \) are suitable), the solution takes the appearance

\begin{equation}
u(t) = G(t, \tau)u(\tau) + \int_{\tau}^{t} G(t, \xi)f(\xi)d\xi.
\end{equation}

Formally the Green's operator \( G(t, \xi) \) may be written \( G(t, \xi) = \exp[ -A(t-\xi) ] \) (for general results in this direction see for example \([1; 2; 3]\)). In this article we propose to study representations related to \( 1.2 \) for solutions of general operational differential equations \( Su = f \) (the operators need not be differential operators of course but therein lies the motivation, see \([4; 5]\); cf. also the papers \([3; 6; 7; 8; 9; 10]\)).

2. Basic framework. Let \( H \) be a Hilbert space and \((S_0, S_0')\) a formally adjoint pair of closed densely defined operators in the sense

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of Browder [7]. Define $S_i = S_i^*$ (then $S_0 \subseteq S_1$) and let $H_0 = D(S_0)$, $H_1 = D(S_1)$, where $H_0$ and $H_1$ have the graph topology. Then $H_0 \subseteq H_0 \subseteq H$ (algebraically and topologically) and following [10] we set $H_i = H_0 \oplus B$ where $B$ is the so-called Cauchy space or space of abstract boundary conditions (see [7; 9; 10]). The symbol $\oplus$ denotes here an orthogonal direct sum (topological); when we wish to speak of a not necessarily orthogonal direct sum (topological) of two closed complementary subspaces $M_1$ and $M_2$ of a Hilbert space $M$ we will write $M = M_1 + M_2$ (see here [11, p. 482]). It will be assumed throughout that $S_0$ is 1-1 with $S_0^{-1}$ continuous and that $S_1$ is onto $H$. (Such hypotheses are verified in many problems of interest; they imply (see [7]) that $S_0'$ has a closed range $\mathcal{R}(S_0')$ and that $(S_0, S_0')$ has a solvable realization operator $\mathcal{S}$; $R(S_0)$ is clearly closed also.) Now we will call any topological supplement of $H_0$ in $H_1$ a Cauchy space $Y$ and write $H_1 = H_0 + Y$ where in general $H_0$ and $Y$ are not orthogonal. Clearly any such $Y$ is isomorphic to $B$ (both are isomorphic to $H_0/H_0$). Then operators $\mathcal{S}$ such that $S_0 \subseteq \mathcal{S} \subseteq S_1$ are characterized by linear subspaces $\hat{\Gamma}$ of $\Gamma$; that is, $\mathcal{H} = D(\mathcal{S})$ is the set $\{u_1: u_1 \in H_1; j u_1 \in \hat{\Gamma} \subseteq \Gamma\}$ where $j: H_1 \to \Gamma$ is the (open) projection determined by $H_0$ and $Y$. Then $\mathcal{H} = H_0 + \hat{\Gamma}$ and $\mathcal{H}$ would be given the graph topology. The following diagram will be useful in illustrating the subject (note ker $S_1$ is closed in $H$ or $H_i$)

\[
\begin{array}{cccccc}
0 & \to & \ker S_1 & \to & 0 \\
\downarrow & & \downarrow & & \downarrow \\
0 & \to & H_0 & \to & H_1 & \to & \Gamma & \to & 0 \\
S_0 & \downarrow & \downarrow & \downarrow & \downarrow \\
R(S_0) & \to & H & \to & 0 \\
\end{array}
\]

The horizontal and vertical sequences are exact (and split by the Banach theorem of homomorphisms). The continuous maps $i$ (injection), $S_1$, and $j$ (projection) may be thought of as morphisms in the category of Hilbert spaces. Note now that $H_0 + \ker S_1$ is closed and hence a topological direct sum since if $u_n$ is Cauchy in $H_0 + \ker S_1$ with $u_n = u_{0n} + u_{1n}$ then $S_0 u_{0n}$ converges which implies that $u_{0n} = u_{0n}$ converges. The diagram (2.1) may be further expanded as follows (cf. [7]), defining $\hat{\Gamma}$ to be any topological supplement of $H_0 + \ker S_1$ in $H_1$. 

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\[
H_0 + \ker S_1 + \Gamma \xrightarrow{j} \{0\} + \Gamma_0 + \Gamma
\]
(2.2)
\[
R(S_0) + \{0\} + \mathcal{H}
\]
where \(\Gamma_0 = j(\ker S_1)\), \(\mathcal{H} = S_1\Gamma\), and in abuse of notation we identify \(\Gamma\) and \(j\Gamma\). It is clear that \(\mathcal{H}\) is closed since \(S_1\) is open and an isomorphism on \(\Gamma\); to see that \(\mathcal{H} \cap R(S_0) = \{0\}\), suppose the contrary. Thus if \(h_0 \in H_0\), \(h \in \Gamma\), and \(S_0h_0 = S_1h\), it follows that \(h_0 - h \in \ker S_1\); since \(h_0 - h \in H_1 - \ker S_1\) we must have \(h_0 - h = 0\). Evidently \(H = R(S_0) + \mathcal{H}\).

We define the Green’s operator to be the map \(\mathcal{G}: (j\mu_1, S_1\mu_1) \rightarrow \mu_1: \Gamma \times H \rightarrow H_1\) which recovers \(\mu_1\) from a knowledge of \(j\mu_1\) and \(S_1\mu_1\). It is seen from the diagrams that \(\mathcal{G}\) is well defined (if \(j\mu_1 = S_1\mu_1 = 0\) then \(\mu_1 \in \ker S_1 \cap H_0 = \{0\}\)). Moreover suppose \(j\mu_1 \rightarrow 0\) in \(\Gamma\) and \(S_1\mu_1 \rightarrow 0\) in \(H\); then writing \(\mu_1 = \mu_0 + \mu\), \(\mu_0 \in H_0\), \(\mu \in \Gamma\), we have \(\mu \rightarrow 0\) in \(\Gamma\) and \(S_1\mu + S_0\mu_0 \rightarrow 0\) in \(H\). Hence \(\mu \rightarrow 0\) in \(H\), \(S_1\mu \rightarrow 0\) in \(H\), and \(S_1\mu + S_0\mu_0 \rightarrow 0\) in \(H\). This implies \(S_0\mu_0 \rightarrow 0\) in \(H\) and therefore \(\mu_0 \rightarrow 0\) in \(H\) by the continuity of \(S_0^{-1}\). Thus finally \(\mu_1 \rightarrow 0\) in \(H_1\) and we have

Proposition 1. The map \(\mathcal{G}: \Gamma \times H \rightarrow H_1\) defined by \(\mathcal{G}(j\mu_1, S_1\mu_1) = \mu_1\) is continuous.

It should be noted that \(\mathcal{G}\) is not a bilinear map in the usual sense and is defined only on the set \(G = \{(j\mu_1, S_1\mu_1)\} \subset \Gamma \times H\).

3. Decomposition of the Green's operator. By the preceding it follows that if \(j\mu_1 = 0\) (i.e. \(\mu_1 \in H_0\)) then \(\mathcal{G}(0, S_1\mu_1)\) defines a continuous map \(\mathcal{G}_2: R(S_0) \rightarrow H_1\). Clearly on \(R(S_0)\), \(\mathcal{G}_2\) may be written as \(S_0^{-1} = S^{-1}\) where \(S\) is a solvable realization operator for \((S_0, S_1)\); hence \(\mathcal{G}_2\) may be extended (as \(S^{-1}\)) to a continuous map \(\mathcal{G}_2: H \rightarrow H_0 + \Gamma\) (cf. [7]). On the other hand if \(\mu_1 \in \ker S_1\), then \(\mathcal{G}(j\mu_1, 0)\) determines a continuous map \(\mathcal{G}_1: T_0 \rightarrow H_1\) (the identity) which may be extended to a continuous map (the identity) \(\mathcal{G}_1: \Gamma_0 + \Gamma \rightarrow H_1\). Then for \(\mu_1 \in H_0 + \ker S_1\)

\[
\mu_1 = \mathcal{G}_1(j\mu_1) + \mathcal{G}_2(S_1\mu_1),
\]
(3.1)
whereas for \(\mu_1 \in \Gamma\) we must have

\[
\mu_1 = \mathcal{G}_1(j\mu_1) = \mathcal{G}_2(S_1\mu_1).
\]
(3.2)

Our interpretation of (1.2) is

\[
\mu_1 = \mathcal{G}_2(\rho S_1\mu_1) + \mathcal{G}_1(j\mu_1),
\]
(3.3)
where \(\rho: H \rightarrow R(S_0)\) is the projection, determined by \(R(S_0)\) and \(\mathcal{H}\). Another formula for the solution similar to (3.3) is
\( u_1 = \mathcal{G}_2(S_1 u_1) + \mathcal{G}_1(\beta_j u_1), \)

where \( \beta : \Gamma \to \Gamma_0 \) is the projection, determined by \( H_0 \) and \( \ker S_1 \). Note that the split \( H_0 + \ker S_1 \) is predetermined; however there is still liberty in choosing \( \Gamma \) and hence \( \tilde{H} \).

We recall now the notion of a kernel for an operator \( T : \mathcal{C} \to \mathcal{C}_1 \) (see here for example \([12; 13; 14]\)); we consider kernels in the sense of Aronszajn and will not attempt to treat here situations requiring the Schwartz kernel theorem (see \([15]\)). Assuming \( \mathcal{C} \) and \( \mathcal{C}_1 \) are separable Hilbert spaces of equivalence classes of measurable functions over a regular measure space \((X, \mu)\) (see \([12]\)), then \( T \) has a kernel \( T(y, \cdot) \) if:

1. for all \( y \in X \), \( T(y, \cdot) \in \mathcal{C} \);
2. the map \( y \to T(y, \cdot) : X \to \mathcal{C} \) is measurable;
3. for all \( h \in D(T) \), \( (Th)(y) = (h, T(y, \cdot)) \) almost everywhere.

If for example all functions in the range of a bounded operator \( T \) are continuous then following Theorem 4 of \([12]\) it is seen that \( T \) has a kernel \( T(y, \cdot) \). This will often prevail in applications (cf. \([17]\)).

Suppose now that \( S_1 \) and \( S_2 \) have kernels \( g_1(t, \cdot) \) and \( g_2(t, \cdot) \); \( S_1 \) and \( S_2 \) are considered as operators in \( \Gamma \) and \( H \) respectively. Then for example (3.3) may be written (see \([16]\) for extensions of (1.2))

\[
(3.5) \quad u_1 = (\rho S_1 u_1, g_2(t, \cdot))_H + (j u_1, g_1(t, \cdot))_H.
\]

We denote the adjoints of continuous maps \( T \) by \( ^*T \) and those of unbounded maps \( T \) by \( T^* \). Then from (3.5), since \( g_1 \in \Gamma \)

\[
(3.6) \quad u_1 = (u_1, ^*S_1 ^*g_2(t, \cdot) + j^*g_1(t, \cdot))u_1.
\]

The following exact sequences indicate how the maps work:

1. \( 0 \to \tilde{H} \to H \xrightarrow{\rho} R(S_0) \to 0; \)
2. \( 0 \to H \ominus R(S_0) \to H \xrightarrow{\iota_0} H \ominus \tilde{H} \to 0; \)
3. \( 0 \to H_0 \to H_1 \xrightarrow{j} \Gamma \to 0; \)
4. \( 0 \to H_1 \ominus \Gamma \to H_1 \xrightarrow{j} H_1 \ominus H_0 \to 0; \)
5. \( 0 \to \ker S_1 \to H_1 \xrightarrow{S_1} H \to 0; \)
6. \( 0 \to H \xrightarrow{^*S_1} H_1 \ominus \ker S_1 \to 0 \)

(note also \( ^*S_1 : R(S_0) \to H_1 \ominus (\Gamma + \ker S_1) \) and \( ^*S_1 : \tilde{H} \to H_1 \ominus (H_0 + \ker S_1) \)).

It is seen that certain problems arise because of the fact that even if \( H_1 = (H_0 + \ker S_1) \oplus \Gamma \) it is not true necessarily that \( H = R(S_0) \oplus \tilde{H} \), where \( \tilde{H} = S_1 \Gamma \). For example if we choose \( \tilde{H} \) first, orthogonal to...
\( R(S_0) \), and define \( \tilde{\Gamma} = S_0 \tilde{H} \), then \( \tilde{\Gamma} \) is orthogonal to \( H_0 + \ker S_1 \); however then \( S_0 \tilde{\Gamma} \neq \tilde{H} \) in general.

**Proposition 2.** Assume \( \xi_1 \) and \( \xi_2 \) have kernels as above; then \( H_1 \) has a reproducing kernel given by

\[
(3.7) \quad h_1(t, \cdot) = \xi_1 \cdot \rho g_2(t, \cdot) + \cdot jg_1(t, \cdot).
\]

We may relate \( \xi_1 \) to our original operators as follows. Assume \( v \in H \) and \( 'S_1 v = w \); then for all \( u \in H_1 \) we have \((S_1 u, v)_H = (u, w)_H\). This means \((S_1 u, v - S_1 w)_H = (u, w)_H\). Therefore \( v - S_1 w \in D(S_0) \) and since \( S_1 \) is an isometry it follows that \( v = S_0 (v - S_1 w) \) (recall \( H_1 \) is dense in \( H \)). Thus \( v \) appears as a solution of the equation \((v - S_1 w) = (S_0)'^{-1}w\).

We note that \( g_1(t, \cdot) \) as defined is a reproducing kernel for \( \Gamma \) and thus for \( u_t \in \Gamma \) there results \( u_t = (u_1, h_1)_H = (u_1, g_1)_H \). In general \( g_1 \) is the component of \( h_1 \) in \( \Gamma \) when \( H_1 \) is written in the form \( \Gamma \oplus (H_1 \ominus \Gamma) \). It should be observed that \( H_0 \) orthogonal to \( \ker S_1 \) in \( H_1 \) is impossible and this fact is closely connected with the development which we have given. A result similar to (3.7) can also be obtained using (3.4). By virtue of the above we may now write (3.7) in a form suitable for calculation.

\[
(3.8) \quad '\rho g_2 = (S_0)'^{-1} + \xi_1 \cdot (h_1 - \cdot jg_1).
\]

This formula will not however entirely determine \( g_2 \) in terms of \( h_1 \) and \( g_1 \); it defines \( g_2 \) up to a term in \( H \ominus R(S_0) \). However, this is sufficient and we have

**Proposition 3.** The component of \( g_2 \) in \( R(S_0) \) is determined by (3.8) if \( h_1 \) and \( g_1 \) are known. If therefore \( \tilde{H} \) is chosen orthogonal to \( R(S_0) \) (with \( \tilde{\Gamma} = \tilde{S}^{-1} \tilde{H} \)), then \( g_2(\rho_{S_0} u_1) \) is fully determined by (3.8).

On the other hand let \( h_1 \) be given; then \( g_1 \) is determined as the component of \( h_1 \) in \( \Gamma \) when \( H_1 \) is decomposed as \( H_1 = \Gamma \oplus (H_1 \ominus \Gamma) \). Thus if \( J \) is the orthogonal projection \( J : H_1 \to \Gamma \) then \( g_1 = J h_1 \). Define then the element \( '\rho g_2 = (S_0)'^{-1}(h_1 - \cdot jg_1) \). This is well-defined since if \( h_1 = h_1 + g_1 \), \( h_1 \in H_1 \ominus \Gamma \), \( g_1 \in \Gamma \), then \( j h_1 = j g_1 = g_1 \in H_1 \ominus H_0 \) and since \( \tilde{J} \) is a projection \( h_1 - \hat{g}_1 \in H_1 \ominus \Gamma \); thus \( h_1 - \hat{g}_1 \in H_1 \ominus \ker S_1 \) with \( 'S_0'^{-1}(h_1 - \hat{g}_1) \) well defined. Now since \( h_1 - \hat{g}_1 \in H_1 \ominus \Gamma \) we have \( 'S_0'^{-1}(h_1 - \hat{g}_1) \in R(S_0) \) and thus \( '\rho g_2 \in R(S_0) \). Assuming now \( \tilde{H} = R(S_0) \oplus \tilde{H} \) with \( \tilde{\Gamma} = \tilde{S}^{-1} \tilde{H} \), it follows that \( '\rho g_2 \) defines an element \( g_2 (= '\rho g_2) \) in \( R(S_0) \) with

\[
(3.9) \quad (\rho_{S_1} u_1, g_2)_H = (S_{1} u_1, '\rho g_2) = (S_{1} u_1, 'S_0'^{-1}(h_1 - \cdot jg_1)) = (u_1, h_1 - \cdot jg_1) = (u_1, g_1) = g_2(\rho_{S_1} u_1).
\]

Hence \( g_2 \) has a kernel \( g_2 \) in \( R(S_0) \) given by \( '\rho^{-1} 'S_0'^{-1}(h_1 - \cdot jg_1) \).
Proposition 4. Assume \( H_1 \) has a reproducing kernel \( h_1 \) and \( H = R(S_0) \oplus \hat{H} \). Then \( g_2 \) has a kernel in \( R(S_0) \) determined by (3.8).

Added in proof. The results of this paper are used in constructing abstract Green's operators in [16] for problems related to [5]. It is shown that \( S = S^* \) (notations of [5]) and formulas such as (3.8) and (1.2) can be studied in more detail.

Bibliography


Rutgers, The State University