THE SHARPNESS OF SARIO'S GENERALIZED PICARD THEOREM

BURTON RODIN

We present here the example referred to in L. Sario [1; 2] which establishes the sharpness of his extensions of Picard's theorem. The nonintegrated estimate for the number of Picard values appears in [1], the integrated estimate in [2]. We use the notation introduced in these two papers.

The Riemann surface $W$ will be described as an $n$-sheeted ramified covering of the nonextended $z$-plane. Consider $n$ copies of this plane slit along the rays $\{\text{Re } z<0, \text{Im } z=2\pi i h\}_{h=0,\pm 1,\ldots}$. Among these sheets identify the edges of the slits which belong to the same value of $h$ so that $2\pi i h$ is a branch point of multiplicity $n$. The resulting surface has the capacity function $p=n^{-1} \log |z|$. The meromorphic function

$$w = \left(\frac{e^{+\pi i}}{e^{+\pi i} + 1}\right)^{1/n}$$

is admissible. It omits the origin and the $n$ values $e^{2\pi i k/n}$. The poles are at $z=2\pi ih$. $W_m$ is the set $\{|z| \leq e^{2\pi n m}\}$ lifted to $W$. Let $\Delta$ be a small neighborhood of $\infty$ in the $w$-plane. Using Hurwitz' formula one obtains

$$e_m^+ \sim (n - 1)e^{2\pi n m}/\pi,$$

$$n_m(\Delta) \sim e^{2\pi n m}/\pi,$$

$$\epsilon = \lim sup \frac{e^+}{S} \leq \lim sup \frac{e^+}{n_m(\Delta)} = n - 1.$$

But $\epsilon+2$ is never less than the number $P$ of Picard values which is $n+1$. Consequently $\epsilon = n - 1$ and the nonintegrated form of the generalized Picard theorem is sharp for every $P \geq 2$.

The computations for the integrated form give

Received by the editors February 27, 1963.

1 Supported by the Air Force Office of Scientific Research and the U. S. Army Research Office (Durham) under contract no. DA-04-495-ORD-1959.

2 This example appeared in the author's doctoral dissertation submitted to the University of California, Los Angeles.

373
\[ E(k) \sim (n - 1)e^{2\pi n k}/2\pi^2 n, \]
\[ N(k, \infty) \sim e^{2\pi n k}/2\pi^2 n, \]
\[ \eta = \lim \sup E(k)/T(k) \leq \lim E(k)/N(k) = n - 1. \]

It follows that the integrated estimate \( P \leq 2 + \eta \) is also sharp for \( P \geq 2 \).

References
