SOME HAUSDORFF MATRICES NOT OF TYPE M

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Let \( A = (a_{nk}) \) denote an infinite matrix. Then \( A \) is said to be of type \( M \) if the conditions

\[
\sum_{n=0}^{\infty} |\alpha_n| < \infty, \quad \sum_{n=0}^{\infty} \alpha_n a_{nk} = 0 \quad (k = 0, 1, 2, \ldots)
\]

always imply \( \alpha_n = 0 \) \((n = 0, 1, 2, \ldots)\).

Matrices of type \( M \) were first introduced by Mazur [3] and so-named by Hill [2]. In [2], Hill developed several sufficient conditions for a Hausdorff matrix to be of type \( M \). He showed that there exists a regular Hausdorff matrix not of type \( M \). The particular matrix used contained a zero on the main diagonal. He also posed the following question: Does there exist a regular Hausdorff matrix which has no zero elements on the main diagonal and which is not of type \( M \)? The purpose of this note is to answer the above question in the affirmative and establish several other related theorems.

A matrix \( A = (a_{nk}) \) is called triangular if \( a_{nk} = 0 \) for all \( k > n \), and is called a triangle if \( A \) is triangular and \( a_{nn} \neq 0 \) for each \( n \). (Some authors use the word normal instead of triangle.) Throughout this paper all matrices and sequences contain real entries.

If we use the words finite sequence to describe a sequence containing only a finite number of nonzero terms, it is clear that a triangular matrix which is not a triangle cannot be of type \( M \), since a finite sequence can be found satisfying (1). Also, if a matrix is a triangle, there can be no finite sequence as a solution of (1). Hill's example is a triangular matrix, not a triangle.

Let \( \mu = \{\mu_k\} \) be a sequence, \( \Delta \) a forward difference operator defined by \( \Delta \mu_k = \mu_k - \mu_{k+1} \), \( \Delta^n \mu_k = \Delta(\Delta^{n-1}\mu_k) \), \( n, k = 0, 1, 2, \ldots \). Then a Hausdorff matrix \( H = (h_{nk}) \) is written in the form \( h_{nk} = C_{n,k} \Delta^{n-k} \mu_k \) for \( k \leq n \), and \( h_{nk} = 0 \) for \( k > n \). The sequence \( \mu \) is called the generating sequence for the matrix \( H \), and, for a regular matrix, we have the representation

\[
\mu_n = \int_0^1 u^n d\nu(u) \quad (n = 0, 1, 2, \ldots),
\]

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where \( q(u) \) is a function of bounded variation on \( 0 \leq u \leq 1 \), \( q(0+) = q(0) = 0, q(1) = 1 \), and \( q(u) = \frac{q(u+0) + q(u-0)}{2} \) for \( 0 < u < 1 \). For other properties of Hausdorff matrices see [1, XI]. The function \( q(u) \) is commonly referred to as the mass function for \( \mu \).

In [2] it has been shown that, for regular Hausdorff matrices, (1) can be written in the form

\[
\int_0^1 g_k(u) dq(u) = 0 \quad (k = 0, 1, 2, \ldots),
\]

where

\[
g_k(u) = \sum_{n=0}^{\infty} a_n C_n u^k (1-u)^{n-k} \quad (k = 0, 1, 2, \ldots).
\]

Each of the functions \( g_k(u) \) represents an absolutely and uniformly convergent series on \( 0 \leq u \leq 1 \), and, for every \( k \),

\[
g_k(u) = (-1)^k u^k q(u)/k!
\]

**Theorem 1.** Let

\[
\mu_n = \frac{b(n-a)}{(-a)(n+b)}, \quad a > 0, b > 0 \quad (n = 0, 1, 2, \ldots).
\]

Then the corresponding regular Hausdorff matrix is not of type M.

**Proof.** If \( a \) is a positive integer, then \( H \) is not of type M as remarked above, since it has a zero on its diagonal.

Assume \( a \) is not a positive integer. We may write \( \mu_n \) in the form

\[
\mu_n = \frac{-b}{a} + \frac{(a+b)}{a} \left( \frac{b}{n+b} \right),
\]

and the corresponding mass function is

\[
q(u) = \begin{cases} 
1, & u = 1, \\
(a+b)u^n/a, & 0 \leq u < 1.
\end{cases}
\]

Choose \( \alpha_0 = 1, \alpha_n = (-1)^n a(a-1) \cdots (a-n+1)/n! \), \( n \geq 1 \). Then \( \sum_{n=0}^{\infty} |\alpha_n| < \infty \), and from (3) \( g_0(u) = \sum_{n=0}^{\infty} \alpha_n (1-u)^n = u^a \). Using (4), we obtain \( g_k(u) = (-1)^k a(a-1) \cdots (a-k+1)u^k/k! \), and the \( g_k(u) \) satisfy (2). Therefore \( H \) is not of type M.

The concept of type M is of value only for reversible matrices, in particular, triangles. In general, it is superseded by the concept perfect; namely, that convergent sequences lie densely in the set of
sequences transformed into convergent sequences by the matrix. For reversible matrices perfect and type M are equivalent.

Since Hill's example is not reversible, it is reasonable to ask if it is perfect. The Hausdorff matrix in the example has moment generating sequence

\[ \mu_n = \frac{6(n - 1)^2}{(n + 1)(n + 2)(n + 3)}, \quad n = 0, 1, 2, \ldots, \]

with corresponding mass function \( Q(u) = 16u^3 - 27u^2 + 12u, 0 \leq u \leq 1. \)

Instead of the corresponding Hausdorff matrix \( H, \) consider the matrix \( K, \) which agrees with \( H \) except that \( k_{11} = 1. \) Then \( H \) and \( K \) have the same convergence domains. Since \( K \) is a triangle, \( K \) is perfect if and only if it is of type M. Furthermore, if one chooses \( \alpha_0 = \alpha_1 = 0, \) \( \alpha_n = 1/n(n - 1) \) for \( n \geq 2, \) then equations (3) and (4) are applicable, and

\[ g_0(u) = \sum_{n=2}^{\infty} \frac{(1 - u)^n}{n(n - 1)}. \]

Term by term differentiation leads us to

\[ g_0''(u) = \sum_{n=2}^{\infty} (1 - u)^{n-2} = \frac{1}{u}. \]

Noting that \( g_0'(1) = g_0(1) = 0, \) we obtain \( g_0(u) = u(\log u - 1). \) Using (3), the conditions in (2) become

\[ \int_0^1 u \, dQ = 0, \]
\[ \int_0^1 u \log u \, dQ = 0, \]

and

\[ \int_0^1 (u \log u - u) \, dQ = 0. \]

It is easy to show that the first two are satisfied. The third condition is automatically satisfied, since it is a linear combination of the first two.

Therefore \( K \) is not of type M, and \( H \) is not perfect.

If one examines the sequences of Theorem 1 for \( a \) an integer, then one notes that the corresponding matrices are not reversible. It remains to determine if each such matrix is perfect.
Theorem 2. Let
\[ \mu_n = \frac{b(n - r)}{(-r)(n + b)}, \quad a > 0, \ r \ a \ positive \ integer \ (n = 0, 1, 2, \cdots). \]

Then the corresponding regular Hausdorff matrix is not perfect.

Proof. The technique will be the same as that of the preceding example; that is, let \( K \) be \( H \) with \( k_r = 1 \). It remains to show that \( K \) is not of type M. For all \( k > r \), equations (3) and (4) are valid. We wish to determine a sequence \( \alpha \) which will satisfy (2). Using (5), (2) becomes
\[ 0 = \int_0^1 g_k(u) dq = \frac{b}{r} \left[ -\alpha_k + (r + b) \sum_{n=k}^\infty \alpha_n \frac{\Gamma(n + 1) \Gamma(k + b)}{\Gamma(k + 1) \Gamma(n + b + 1)} \right], \]
for \( k = r + 1, r + 2, \cdots \). Solving the above system for the \( \alpha_k \)'s we obtain the recursion formula
\[ \alpha_{k+1} = \frac{(k - r)\alpha_k}{k + 1}, \]
which gives us
\[ \alpha_{r+m+1} = \frac{m!\alpha_{r+1}}{(r + 2)(r + 3) \cdots (r + m + 1)}, \quad m = 1, 2, 3, \cdots. \]

Selecting \( \alpha_{r+1} = 1 \), we can show that \( \sum_{m=1}^\infty |\alpha_{r+m+1}| < \infty \). Substituting in (3) we obtain \( g_{r+1}(u) = u^r \). Using (4) leads to
\[ g_0^{(r+1)}(u) = \frac{(-1)^{r+1}(r + 1)!}{u}, \]
and hence that
\[ g_k(u) = \frac{u^r}{\binom{k}{m-1}}, \quad k = r + 1, r + 2, \cdots. \]

Thus conditions (2) reduce to showing that
\[ \int_0^1 u^r dq = 0, \]
a condition easily verified.

There now remains the problem of determining the values \( \alpha_0 \) through \( \alpha_r \). Because \( H \) has been modified in row \( r \), it is not possible
to use (6) to obtain $g_0^{(r)}(u)$ and hence $g_0(u)$. However, one can use (1) to determine $\alpha_r$. Continued use of (1) will determine the remaining values $\alpha_{r-1}, \cdots, \alpha_0$. Thus $K$ is not of type $M$, and $H$ is not perfect.

The following is an open question. Does there exist a regular Hausdorff matrix which is perfect, but not of type $M$?

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References


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