CONVEX POLYGONS

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1. Introduction and terminology. In the two-dimensional Euclidean plane $E_2$, if $\gamma$, $\delta$ are any linearly independent vectors, define projections $P$, $Q$ by $P(x\gamma+y\delta) = x\gamma$, $Q(x\gamma+y\delta) = y\delta$. Denote by $L$, $M$ the lines consisting respectively of the set of points $\{a\gamma\}$ for all real $a$, and of the set of points $\{b\delta\}$ for all real $b$. A convex region $K$ in $E_2$ is of standard type, by projections parallel to the directions of $\gamma$, $\delta$, in case there is a translate $C$ of $K$, and a choice of vectors $\gamma$, $\delta$, such that the projections $PC$, $QC$ of $C$ coincide respectively with the sections $L \cap C$, $M \cap C$ of $C$ with the lines $L$, $M$.

For example, any rectangle is of standard type by projections $P$, $Q$ parallel to its diagonals; it is also of standard type by projections parallel to its sides. A triangle is of standard type by projections $P$, $Q$ parallel to a side and to any chord joining the opposite vertex to a point of the side.

By saying that a region $C$ is symmetric, we mean that it can be so translated in $E_2$ that it is symmetric or centered in the origin $\theta$ of $E_2$; that is, whenever $(x, y) = \sigma = x\gamma + y\delta$ is a point of $C$, so is $-\sigma = (-x, -y)$. In dealing with a symmetric region, we always assume that it is so located in $E_2$ that its center is at $\theta$.

We are now able to state the principal results of this paper. Every closed symmetric convex polygon $C$ is of standard type, both by projections parallel to a pair of diagonals of $C$, and by projections parallel to a pair of sides of $C$. Each nonsymmetric closed convex polygon $D$ has an associated symmetric polygon $C$, with sides and diagonals parallel to those of $D$. Therefore by the results for symmetric polygons, $D$ also is of standard type, by projections parallel to a pair of sides, and by projections parallel to a pair of diagonals. By polygonal approximations, this implies that every bounded planar convex region is of standard type. In the final section, various functional analytic implications are derived, and open questions and a conjecture are indicated.

2. Diagonal projections of symmetric convex polygons. For any symmetric closed convex polygon $C_p$ having $2p$ sides, denote by $\alpha_1, \ldots, \alpha_p$ the angles, measured from a horizontal ray from $\theta$, of the
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sides 1, ⋅⋅⋅ , p in counterclockwise succession, and by β₀, ⋅⋅⋅ , βᵢ₋₁, the angles of the diagonal radii from θ to the initial points of sides 1, ⋅⋅⋅ , p. A rectangle is of standard type by projections parallel to its diagonals; we shall now prove by induction that Cᵢ is of standard type by projections parallel to a pair of its diagonals.

Let V₁, ⋅⋅⋅ , Vᵢ denote the vertices (terminal points of sides 1, ⋅⋅⋅ , p) of Cᵢ. Then vectors to the opposite vertices — V₁, ⋅⋅⋅ , — Vᵢ of Cᵢ have respective angles β₁ ± π, ⋅⋅⋅ , βᵢ₋₁ ± π, βᵢ. If the pair of sides i, i + 1 and the opposite pair of sides are replaced by single sides joining Vᵢ₋₁ to Vᵢ₊₁ and — Vᵢ₋₁ to — Vᵢ₊₁, we obtain a symmetric convex polygon Cᵢ₋₁, which has 2(p − 1) sides. The induction hypothesis is that every Cᵢ₋₁ if of standard type by projections parallel to a pair of diagonals of Cᵢ₋₁.

If the projections by which Cᵢ₋₁, i are of standard type are in directions βₖ, k < i − 1, and βₖ, j > i + 1, then in consequence of convexity of Cᵢ, Cᵢ is of standard type by the same projections. In case k = i − 1, Cᵢ is of standard type by the same projections provided that βᵢ ≤ αᵢ; and in case j = i + 1, Cᵢ is of standard type by the projections provided that αᵢ₊₁ ≤ βᵢ₊₁. Either or both inequalities are satisfied, if they need to be for the induction, at a vertex Vᵢ such that αᵢ₊₁ ≤ βᵢ₋₁ + π and βᵢ₊₁ ≤ αᵢ, where αᵢ₊₁ = αᵢ + π, β₀ = β₀ + π, βᵢ₊₁ = βᵢ₊₁ + π.

In case Cᵢ is such that there is no vertex Vᵢ at which both inequalities of the above pair are satisfied, so that we cannot conclude that Cᵢ is of standard type by the projections for some Cᵢ₋₁, i, then for each i = 1, ⋅⋅⋅ , p, we have that either

(1) βᵢ₊₁ > αᵢ

or

(2) αᵢ₊₁ > βᵢ₋₁ + π

or both, are satisfied. For any i, by convexity αᵢ₊₁ > αᵢ, and βᵢ₋₁ + π > αᵢ > βᵢ. Consider, for example, vertices V₁ and V₂. In case say β₀ > α₀ and α₀ > β₁ + π, then also β₀ + π > α₀ > β₁, so in this case Cᵢ is of standard type by projections in directions β₁, βᵢ.

For any possible Cᵢ as in the preceding paragraph, we may indicate which of the inequalities (1), (2) is satisfied at each vertex V₁, ⋅⋅⋅ , Vᵢ, by a binary symbol having p digits, such as 121 ⋅⋅⋅ 1. Digit 1 indicates that (1) is satisfied, digit 2, that (2) is satisfied. (If both are satisfied at Vᵢ, the choice between digits 1 and 2 for the kth place is arbitrary.) If the binary symbol for a Cᵢ contains anywhere a successive pair of digits 12, or if it begins with 2 and ends with 1, then Cᵢ is similar to the example of the preceding paragraph,
and therefore is of standard type by projections parallel to a successive pair of diagonals.

The only remaining cases are those for which the binary symbol consists of either all 1's, or all 2's. We now complete the proof by showing that these cases for \( C_p \) are impossible. The case of all 2's becomes the case of all 1's if the numeration and positive sense of angles are changed from counterclockwise to clockwise (or if the plane of \( C_p \) is inverted), so it is sufficient to dispose of the case of all 1's. We therefore assume that \( \beta_{i+1} > \alpha_i \), \( i = 1, \ldots, p \). By convexity, there is a vertex \( V_k \) such that a line at angle \( \beta_0 \) through \( V_k \) is a support line of \( C_p \). By affine transformation, we may suppose that the difference \( \beta_k - \beta_0 \) is \( \pi/2 \), and that \( \beta_0 = 0 \). In case \( 1 < k \leq p - 1 \), if \( v_1 \) is the vertical projection of side 1, we have \( d_0 < v_1 \cot \beta_1 \), where \( d_0 \) is the length of the horizontal diagonal chord. If \( d_k \) is the length of the vertical diagonal chord, by hypothesis we have \( d_k > v_1 \), and \( d_0 > d_k \cot(\alpha_p - \pi) \). But since \( \beta_{i+1} = \beta_i + \pi > \alpha_i \), we have \( \cot \beta_i < \cot(\alpha_i - \pi) \), \( d_0 > v_1 \cot(\alpha_i - \pi) > v_1 \cot \beta_i \), which contradicts \( d_0 < v_1 \cot \beta_i \). If \( k = 1 \) and \( p > 3 \), then \( \alpha_2 > \pi, \beta_2 > \alpha_2 \), but since \( \beta_0 = 0 \), we have \( \beta_2 < \pi \), which is a contradiction. If \( k = 1 \) and \( p = 3 \), then by hypothesis \( \alpha_3 > \pi \), and also \( \beta_3 = \beta_0 + \pi = \pi > \alpha_2 \), which again is a contradiction. Therefore the case of all 1's is impossible, and it has been demonstrated fully that every \( C_p \) is of standard type by projections parallel to some pair of its diagonals.

3. Duality of projections parallel to sides and diagonals. Points and lines in \( E_2 \) are related to lines and points in the dual plane \( E_2^* \) of \( E_2 \) by means of the basic equation

\[
(3) \quad x\xi + y\eta - 1 = 0.
\]

A point \((x, y)\) of \( E_2 \) is dual to the line of points whose coordinates \((\xi, \eta)\) in \( E_2^* \) satisfy equation (3); the line of \( E_2 \) given by (3) for fixed \((\xi, \eta)\) is dual to the point \((\xi, \eta)\) in \( E_2^* \). As usual the finite planes \( E_2, E_2^* \) may be enlarged to be projective planes by adjunction of ideal points at infinity. Then \( \theta \) is dual to the line at infinity of \( E_2^* \); each line \( x \cos \omega + y \sin \omega = 0 \) through \( \theta \) in \( E_2 \) is dual to the point at infinity in the direction \( \omega \) of \( E_2^* \).

The dual points \( U_1, \ldots, U_p, \) and \( -U_1, \ldots, -U_p, \) of the sides \( 1, \ldots, p \), and of the parallel opposite sides, are vertices of the dual polygon \( C_p^* \) of \( C_p \). It is easily shown that \( C_p^* \) is symmetric and convex. If \( C_p \) is of standard type by projections parallel to its diagonals at angles \( \beta_0, \beta_p \), denote by \( B_0, B_p \) the lines containing the corresponding diagonals, and by \( B_0', B_p' \) the parallel lines respectively through vertices \( V_j, V_k \). The dual point \( B_p^* \) is a point of side \( U_j U_{j+1} \) of \( C_p^* \);
since $B_h$ passes through $\theta$, the dual point $B_1\ast$ is a point at infinity. Since $B_h$ and sides $h, h+1$ are copunctal, the points $B_i\ast, U_h, U_{h+1}$ are collinear; that is, the line through $B_i\ast$ in the direction of $B_i\ast$ is parallel to side $U_h U_{h+1}$. Similarly the line through the point $B_i'\ast$ on side $U_h U_{h+1}$ in the direction of the point at infinity $B_i'\ast$ is parallel to side $U_i U_{i+1}$. Therefore $C_p\ast$ is of standard type by projections parallel to sides $U_h U_{h+1}$ and $U_i U_{i+1}$.

Any closed symmetric convex polygon may be taken as a $C_p\ast$; then since $C_p\ast\ast=C_p$ is of standard type by projections parallel to a pair of diagonals, $C_p\ast$ is of standard type by projections parallel to a pair of sides. Direct proof that every $C_p$ is of standard type by projections parallel to a pair of sides seems to be more difficult; or at least the author, after some effort, abandoned his attempt to find a direct proof, and succeeded in finding the preceding proof by appeal to duality.

4. Nonsymmetric convex polygons. If $D$ is any closed convex polygon having $n$ sides, a direction of angular reference may be chosen so that if $\alpha_1, \ldots, \alpha_n$ are the angles of the sides of $D$, then $0<\alpha_1<\ldots<\alpha_n<2\pi$, where the sides are taken in counterclockwise succession. Let $A_1, \ldots, A_n$ be vectors for the sides of $D$, in corresponding counterclockwise directions around $D$. A diameter of $D$ is a chord which joins the points of contact of parallel support lines of $D$.

We may associate with $D$ a symmetric convex polygon $C$ as follows: $C=D-D$ is the set of all vectors of the form $(X-Y)$, where $X \subseteq D, Y \subseteq D$, with respect to an arbitrary origin $\theta$. (P. C. Hammer has named the symmetric convex body $C$ so associated with any convex body $D$ the symmetroid of $D$; the concept also occurs in work of V. L. Klee.)

It may be easily shown that the vectors for the sides of $C$ are $\pm A_1, \ldots, \pm A_n$, where the angle of $-A_k$ is $(\alpha_k-\pi)$ if $\alpha_k \geq \pi$, $(\alpha_k+\pi)$ if $\alpha_k < \pi$, and the sides $\pm A_1, \ldots, \pm A_n$ are arranged in order of increasing angles of the vectors. Diagonals of $D$ which are diameters correspond to diagonal chords of $C$; diameters from a vertex to a side of $D$ correspond to diametral chords between a pair of parallel sides of $C$. Therefore obviously since $C$ is of standard type by projec-
tions parallel to a pair of (nonparallel) sides, $D$ is of standard type by projections parallel to the corresponding sides of $D$; similarly $D$ is of standard type by projections parallel to some pair of diametral diagonals of $D$.

Let a side $A_k$ of $D$ such that $(\alpha_{k+1} - \alpha_{k-1}) \geq \pi$, if any, be called an obtuse side. Then clearly if $D$ has an obtuse side, it is like a triangle, and is of standard type by projections parallel to the side and to any ray at an angle $\beta$ such that $\alpha_{k-1} \leq \beta \leq \alpha_{k+1} - \pi$. All sides of a triangle are obtuse; any convex skew quadrilateral must have an obtuse side; a convex pentagon may have all sides nonobtuse. Since a convex polygon $D$ having no obtuse side is of standard type by projections parallel to a pair of interior diagonals, for such a $D$ there always exists a diametral diagonal which is an obtuse side for two polygons into which $D$ is divided by the diagonal; the two polygons are simultaneously of standard type by projections parallel to the diagonal and to the segments into which the other diagonal is divided.

For any vertex $V_i$ of a convex polygon $D$, consider particular chords from $V_i$ as follows. In case $D$ has an odd number of sides, the chords (from $V_i$ to an interior point of a side of $D$) are those which divide $D$ into two polygons with the same number of sides. In case $D$ has an even number of sides, the chord is the diagonal (from $V_i$ to another vertex of $D$) which divides $D$ into two polygons with the same number of sides. If such a chord is a diameter of $D$, then the two polygons into which $D$ is divided are each of standard type by projections parallel to the chord and to the support lines of the end points of the chord (although $D$ itself need not be of standard type by those projections). Call a vertex $V_i$, such that the particular chords or chord as described are diameters of $D$, a regular vertex. If $V_i$ is regular, the triangle whose sides are extensions of the sides of $D$ adjacent to $V_i$, and whose base is in the support line at the other end of the diameter from $V_i$, circumscribes $D$.

**Theorem.** Every convex polygon $D$ has at least one regular vertex.

**Proof.** Suppose first that the number $n$ of sides of $D$ is odd. By definition, for $j > k$, $V_j$ is regular in case $(\alpha_j - \alpha_{j-k}) < \pi$, and $(\alpha_{j+1} - \alpha_{j-k}) > \pi$, where $\alpha_{n+1} = (\alpha_1 + 2\pi)$. For $j \leq k$, $V_j$ is regular in case $\alpha_j - (\alpha_{j+k+1} - 2\pi) < \pi$, and $\alpha_{j+1} - (\alpha_{j+k+1} - 2\pi) > \pi$. Thus assuming that every vertex is not regular, we have

$$\text{(4)} \quad (\alpha_j - \alpha_{j-k}) \geq \pi, \text{ or } (\alpha_{j+1} - \alpha_{j-k}) \leq \pi,$$

for $j > k$, and
(5) \( (\alpha_{j+k+1} - \alpha_j) \leq \pi, \) or \( (\alpha_{j+k+1} - \alpha_{j+1}) \geq \pi, \)

for \( j \leq k. \) For each \( j, \) by convexity and since \( D \) has \( n \) sides, not both inequalities in (4) or in (5) can be satisfied. Assume that the first member of (4) is true for \( j = k + 1; \) the second member, which is the same as the first member of (5) for \( j = 1, \) then is false. The second member of (5) for \( j = 1, \) which is the same as the first member of (4) for \( j = k + 2, \) then is true. The second member of (4) for \( j = k + 2, \) which is the same as the first member of (5) for \( j = 2, \) then is false. Continuing in this way, we see that on the basis of the assumption that the first member of (4) is true, all the first members of (5) are false, and all the first members of (4) are true. In particular for \( j = 2k + 1 = n, \) \( (\alpha_n - \alpha_{n-k}) \geq \pi, \) and so the second member of (4), namely \( \alpha_{k+1} - \alpha_{k+1} \leq \pi, \) is false; that is \( (\alpha_n + 2\pi) - \alpha_k + 1 > \pi, \) or \( \alpha_k + 1 - \alpha_1 < \pi. \) But this contradicts the assumption that the first member of (4) for \( j = k + 1, \) namely \( \alpha_k + 1 - \alpha_1 \geq \pi, \) is true. Similarly if we assume that the first member of (4) for \( j = n \) and the intervening inequalities, we obtain a contradiction. Therefore, we have now established that \( D \) must always have at least one regular vertex.

For the case of even \( n, n = 2k, \) a vertex \( V_j \) is regular iff \( (\alpha_j - \alpha_{j-k}) \leq \pi \) and \( (\alpha_{j+1} - \alpha_{j-k}) > \pi; \) that is, in case there is a triangle \( T \) circumscribing \( D, \) such that \( V_j \) and extended sides \( j, j + 1 \) are a vertex and adjacent sides of \( T, \) and the opposite vertex \( V_{j-k} \) is on the third side of \( T. \) Consideration of the similar system of alternative inequalities to (4) and (5) yields the conclusion that \( D \) has at least one regular vertex.

Remarks. In case no side of \( D \) is parallel to a diametral diagonal, then \( D \) is of standard type only by projections parallel to sides, or by projections parallel to diagonals. If \( B_k, B_j \) are the diagonals such that \( D \) is of standard type by projections parallel to \( B_k, B_j, \) and if the origin \( \theta \) in \( E_2 \) is chosen at the point of intersection of \( B_k \) and \( B_j, \) then the dual polygon \( D^* \) is a convex polygon which has two pairs of parallel sides, and \( D^* \) is of standard type by projections parallel to the directions of the pairs of parallel sides. Any convex polygon, which has two pairs of parallel sides, is of standard type by projections parallel to the two directions of the pairs of parallel sides, if and only if there is a diameter parallel to each of the two directions, which joins points of the parallel sides in the other direction. This is a dual condition for \( D \) to be of standard type by projections parallel to a particular pair of diametral diagonals.
5. Functional analytic interpretations and equivalent properties. Any bounded symmetric convex region $C$ in $E_2$ may be taken as the unit cell for a Minkowski or Banach norm on $E_2$. It follows from the result of §2 that basis vectors $\gamma$, $\delta$ may always be chosen so that this norm simultaneously has the properties $\| (x+\Delta x)\gamma + y\delta \| \geq \| x\gamma + y\delta \|$ when $x, \Delta x$ have the same sign, $\| x\gamma + (y+\Delta y)\delta \| \geq \| x\gamma + y\delta \|$ when $y, \Delta y$ have the same sign, and $\| x\gamma + y\delta \| \geq \max(\| x\gamma \|, \| y\delta \|)$. Another equivalent property is that $C$ is intermediate between the convex hull and the Cartesian product of the intervals $L \cap C$, $M \cap C$. (The norm $\| x\gamma + y\delta \| = \| x\gamma \| + \| y\delta \|$ has the convex hull for its unit cell.)

Still another equivalent statement is the following: if $\gamma$, $\delta$ are variable vectors of unit norm, and $\sigma$, $\tau$ are unit tangent or support vectors to the unit cell $C$ at the end points of $\gamma$, $\delta$, in directions such that the couples of vectors $(\gamma, \sigma)$ and $(\delta, \tau)$ have opposite senses, then there always exist $\gamma$, $\delta$ such that $\tau = \gamma$ and $\sigma = \delta$. (The author was unable to establish this directly by application of the Brouwer or other fixed point theorem.) Since a support line exists at the end point of every $\gamma$, we may take $\delta = \sigma$, and assert that a $\gamma$ always exists such that $\tau = \gamma$.

It is a theorem of Banach and Mazur [3, pp. 185–187], that every separable Banach space is isometric with a subspace of the space $(C)$ of continuous functions $z(t)$ on a finite closed interval, where the norm in $(C)$ is given by $\| z \| = \max_t | z(t) |$. In particular, the norm for any two-dimensional Banach space $L_2$ is given by $\| x\gamma + y\delta \| = \max_t | x\gamma(t) + y\delta(t) |$, where $\gamma(t)$ and $\delta(t)$ are any pair of continuous functions which span the subspace $M_2$ of $(C)$ which is isometric with $L_2$.

In case of a Banach space $L_2$ such that the unit cell is a symmetric convex polygon $C$ of $2p$ sides, the functions $\gamma$, $\delta$ may be functions on a discrete set of $p$ or $2p$ elements. If we change the independent variable from $t$ to $u = \arctan \delta/\gamma$, if $x\xi_i + y\eta_i - 1 = 0$, $i = 1, \ldots, 2p$, are the equations of the sides of $C$, then we may set $\gamma(u_i) = \xi_i$, $\delta(u_i) = \eta_i$, where $u_i = \arctan \eta_i/\xi_i$, and $\gamma$, $\delta$ may be extended to be continuous functions on the entire interval $0 \leq u < 2\pi$ by interpolating, for each $i$, the continuous graph between $u_i$ and $u_{i+1}$ which corresponds to the line segment joining the points $(\xi_i, \eta_i)$ and $(\xi_{i+1}, \eta_{i+1})$ in the dual plane (the points of this line segment are dual to all the support lines through vertex $V_i$ in the original plane).

For any unit cell $C$, we may approximate $C$ by a sequence $\{ C_n \}$ of circumscribing symmetric convex polygons, with the number of sides of $C_n$ increasing indefinitely with $n$. Further, we may choose the circumscribing polygons so that from a certain stage on they are all of standard type by projections parallel to a pair of sides which maintain constant directions in the succession of circumscribing polygons.
Thus if \( \gamma_n(u) \), \( \delta_n(u) \) are the extended functions corresponding to one of the polygons \( C_n \), if \( C_n \) is of standard type by projections parallel to sides \( h, k \), then we have \( \max_u \left| x\gamma_n(u) + y\delta_n(u) \right| = \left| x\gamma_n(u_h) + y\delta_n(u_h) \right| = \left| x\gamma_n(u_h) \right| \) for sufficiently small \( y \), so \( \delta_n(u_h) = 0 \), and similarly \( \gamma_n(u_h) = 0 \). Therefore the functions \( \gamma(u) \), \( \delta(u) \) which represent \( C \) have the property that \( \delta(u_1) = 0 \) at a value \( u = u_1 \), where \( \gamma(u) \) is maximum, and simultaneously \( \gamma(u_2) = 0 \) at a value \( u = u_2 \) where \( \delta(u) \) is maximum. This, and similar considerations for any bounded convex \( D \) in \( E_2 \) with \( \theta \) interior to \( D \), yield the following theorem.

**Theorem.** If \( \alpha(u) \), \( \beta(u) \) are any pair of linearly independent continuous functions, periodic with period \( 2\pi \), then there exist linear combinations \( y(u) = a\alpha(u) + b\beta(u) \), \( \delta(u) = c\alpha(u) + d\beta(u) \), such that simultaneously \( \delta(u_1) = 0 \) at a value \( u = u_1 \) where \( \gamma(u) \) is maximum, and \( \gamma(u_2) = 0 \) at a value \( u = u_2 \) where \( \delta(u) \) is maximum. If in particular \( \alpha \), \( \beta \) are continuously differentiable, then there always exist distinct values \( u_1 \), \( u_2 \) such that simultaneously \( \alpha'(u_1)\beta(u_2) = \alpha'(u_2)\beta(u_1) \) and \( \alpha'(u_2)\beta'(u_1) = \alpha'(u_1)\beta'(u_2) \).

This theorem follows since any polygon \( D_n \) is of standard type by projections parallel to a pair of sides, and since the function on \( E_2 \) defined by \( p(x, y) = \max_u \left( x\alpha(u) + y\beta(u) \right) \), for any pair of independent continuous functions \( \alpha \), \( \beta \), is positive homogeneous and subadditive, which implies that \( D = \{(x, y): p(x, y) \leq 1\} \) is convex and has \( \theta \) in its interior. The function defined by \( \|(x, y)\| = \max_u \left| x\alpha(u) + y\beta(u) \right| \), where \( \alpha \), \( \beta \) are such that \( \alpha(u + \pi) = -\alpha(u) \), \( \beta(u + \pi) = -\beta(u) \), is a norm.

A convex region \( K \) in \( m \)-dimensional Euclidean space \( E_m \) is of *standard type* if there exist a translate \( D \) of \( K \), and a choice of basis \( \gamma_1, \cdots, \gamma_m \), such that \( P_jD = L_j \cap D \), where \( P_j(x_1\gamma_1 + \cdots + x_m\gamma_m) = c_j\gamma_j \), and \( L_j \) is the line of points \( \{c\gamma_j\} \) for all real \( c \). The author ventures the following conjecture.

**Conjecture.** Every bounded convex region \( D \) in \( E_m \) is of standard type.

A form of the conjecture, which is weaker for nonsymmetric convex regions, would allow the ranges of the projections and lines \( L_j \) to be translates of the lines \( \{c\gamma_j\} \). By the existence of the associated symmetric convex region, as indicated in §4, to establish the truth of at least the weaker form of this conjecture, it would be sufficient to prove it for bounded symmetric convex regions \( C \) in \( E_m \). One method of attempting to prove it might be to apply the above theorem in some way to a set of \( m \) linearly independent continuous functions \( \alpha_1(u), \cdots, \alpha_m(u) \), having the property \( \alpha_j(u + \pi) = -\alpha_j(u) \), \( j = 1, \cdots, m \), which corresponds to symmetry of \( C \). The unit cell
for the norm \( \| (x_1, \ldots, x_m) \| = \max_u |x_1 \gamma_1(u) + \cdots + x_m \gamma_m(u)| \) is of standard type if there exist distinct values \( u_1, \ldots, u_m \), such that \( \gamma_i(u) \) is maximum at \( u = u_i \), and \( \gamma_i(u_j) = 0 \), \( i \neq j \), for \( j = 1, \ldots, m \).

An unsolved problem concerning separable Banach spaces \( M \) is to show whether or not every such space has a base, that is, a set of elements \( \{ \gamma_j \} \) of \( M \) such that each element \( X \) of \( M \) has a unique expansion \( X = \sum_{i=1}^m x_i \gamma_i \), which converges in norm to \( X \). (See [3, pp. 110–111, and p. 238].) In case \( M \) has a base, the projections \( P_j x = x_j \gamma_j \), although uniformly bounded, may have bounds greater than or equal to 2. If the unit cell \( C \) for a separable Banach space \( M \) is of standard type, then \( M \) has a base \( \{ \gamma_j \} \) with the stronger property that the projections \( P_j \) are all of norm 1.

References


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ON UNIFORM CONNECTEDNESS

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Connectedness of topological spaces can be defined in terms of continuous functions to a discrete space (every continuous function to a discrete space is constant) or to the space of real numbers (every continuous real function has the Darboux property; i.e., the range of the function is an interval). We will consider in this paper similar properties for uniform and proximity spaces obtained by replacing continuous functions by uniformly continuous or equicontinuous functions (a function is equicontinuous iff it takes near sets into near sets).

**Definition 1.** A uniform space \( (X, U) \) is uniformly connected iff every uniformly continuous function on \( X \) to a discrete space is constant.

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