1. Introduction. In this paper we examine one of the results in the theory of the proximity spaces of Efremovic [1]:

A set $X$ with a binary relation “$A$ close to $B$” (written $A \delta B$) is a proximity space if and only if there exists a compact Hausdorff space $Y$ in which $X$ can be topologically imbedded so that

\[(1.1) \quad A \delta B \text{ in } X \text{ if and only if } A \text{ meets } B \text{ in } Y \]

($\overline{A}$ denotes the closure of the set $A$) [2].

This proposition raises the question: Can we characterize the relations $\delta$ for which this result holds under weaker conditions on $Y$? In §4 we give an affirmative answer (Theorem 5.3) using rather mild restrictions on $Y$ and on the imbedding of $X$ in $Y$. This result is essentially a corollary to a fundamental theorem (Theorem 4.2).

2. Symmetric generalized proximity spaces. As in [3] we define a symmetric generalized proximity space (or $P_\delta$-space) to be an abstract set $X$ with a binary operation “$A \delta B$” (a $P_\delta$-relation) on its power set satisfying the following axioms:

(P.1) $A \delta (B \cup C)$ implies that either $A \delta B$ or $A \delta C$.

(P.2) $A \delta B$ implies that $A$ and $B$ are nonvoid.

(P.3) If $A$ meets $B$ then $A \delta B$.

(P.4) $A \delta B$ and $b \delta C$ for all $b$ in $B$ implies that $A \delta C$.

(P.5) $A \delta B$ implies $B \delta A$.

We read the symbols “$A \delta B$” as “$A$ is close to $B$”; and we say that “$A$ is remote from $B$” (in symbols, “$A$ not $\delta B$”) if $A$ is not close to $B$.

(2.1) The following facts are evident: (1) If $A \delta B$, $A \subseteq C$, and $B \subseteq D$, then $C \delta D$. (2) Define

$$A^\delta = \{x \in X : x \delta A\};$$

then in a $P_\delta$-space $(A^\delta) \delta (B^\delta)$ if and only if $A \delta B$.

(2.2) In [3] it is shown that there is a topology induced on every $P_\delta$-space $(X, \delta)$ by the closure operation $A \rightarrow A^\delta$. Moreover, this topology is symmetric: $x$ in $\overline{y}$ implies $y$ in $\overline{x}$ for all points $x, y \in X$. Clearly, every $T_1$ topological space is symmetric.

(2.3) Theorem. Given any symmetric topological space $X$ define $\delta_0$ by:

Received by the editors October 13, 1962 and, in revised form, February 9, 1963.
A δ₀ B if and only if A meets B.

Then δ₀ is a Pᵦ-relation and is compatible with the given topology: x δ₀ B if and only if x ∈ B.

Proof. We derive axioms (P.1) through (P.5) by use of the Kuratowski closure axioms [4]. Axioms (P.1), (P.2), (P.3) and (P.5) are trivial results of the closure axioms and (2.4). For (P.4), note that if for a point b and a set C we have b ∩ C ≠ ∅, then there exists a point c in C such that c ∈ b. By symmetry then b ∈ c ⊆ C. Thus, if A ∩ B ≠ ∅ and b ∩ C ≠ ∅ for every b in B then B ⊆ C and so A ∩ C ≠ ∅. It is now clear, from the above argument, that δ₀ is compatible with the given topology.

(2.5) Theorem. Given a Pᵦ-space (X, δ) and δ₀ defined by (2.4) in terms of the topology induced by δ we have that A δ₀ B implies that A δ B for all subsets A and B of X. Thus δ₀ is the smallest P-relation compatible with the topology in a symmetric topological space.

Proof. The demonstration follows directly from (2.3), (2.1) and (P.3).

3. Clusters. A cluster π from a Pᵦ-space (X, δ) is a class of subsets of X satisfying:

(C.1) A δ B for all A, B ∈ π.

(C.2) A ∪ B ∈ π implies that either A ∈ π or B ∈ π.

(C.3) If B δ A for every A in π, then B ∈ π.

Note that this is the same definition used by Leader [5] in introducing clusters for Efremovic proximity spaces.

(3.1) Theorem. For x, a point in a Pᵦ-space (X, δ), the class πₓ of all subsets of X which are close to x is a cluster from X.

Proof. We must show that πₓ satisfies (C.1), (C.2) and (C.3). For (C.1) suppose A, B ∈ πₓ. Then x δ A and x δ B so that, by (2.5), A δ B. For (C.2) suppose A ∪ B ∈ πₓ. Then x δ (A ∪ B) and, by (P.1), this means that either x δ A or x δ B, that is, either A ∈ πₓ or B ∈ πₓ. For (C.3) suppose that A δ C for every C in πₓ. Since, by (P.3), {x} ∈ πₓ, we have in particular, that A δ x or, A ∈ πₓ.

(3.2) The following facts are easily established. (1) Any cluster π from a Pᵦ-space (X, δ) is closed under the operation of supersets: if π is a cluster from X, A ∈ π, and A ⊆ B, then B ∈ π. (2) If A ∈ π, a cluster from X, and a δ B for every a in A, then B ∈ π. (3) If π and π' are clusters from X and π is a subclass of π', then π = π'. (4) If a point x belongs to a cluster π, then π is just the class πₓ of all subsets A of X such that A δ x. (5) Given a cluster π from a nonvoid Pᵦ-space...
(X, δ) and any subset A of X, then either A ∈ π or X − A ∈ π. (6) Let π be a cluster from (X, δ). If A is a subset of X which meets every member of π, then A ∈ π.

4. Extensions characterized by clusters. We say that a subset X of a topological space Y is regularly dense in Y if and only if given U open in Y and p a point in U there exists a subset E of X with p ∈ E ⊆ U, the closure being taken in Y.

(4.1) Theorem. If X is regularly dense in Y, then X is dense in Y. If Y is regular and X is dense in Y then X is regularly dense in Y.

Proof. Y is open in Y, hence for any point p in Y there exists a subset E of X such that p ∈ E ⊆ X ⊆ Y. Since this is true for any p in Y, we have Y ⊆ X ⊆ Y.

For Y regular, y ∈ Y and U an open set of Y containing y we have the existence of an open set V of Y containing y such that V ⊆ U. Now E = V ∩ X is a subset of X and Ē = Cl(V ∩ X) = V ⊆ U, \(^1\) with the second equality following from the density of X in Y. Thus, y ∈ Ē ⊆ U.

(4.2) Theorem. Given a set X and some binary relation δ on the power set of X, the following are equivalent:

(I) There exists a T₁ topological space Y and a mapping f of X into Y such that fX is regularly dense in Y and

(4.3) A δ B in X if and only if Cl(fA) meets Cl(fB) in Y.

(II) δ is a P₃,relation satisfying the additional axiom,

(P.6) Given A δ B in X there exists a cluster π to which both A and B belong.

Proof. Suppose that (I) holds and define δ by (4.3). (P.1), (P.2), (P.3) and (P.5) are trivial consequences of the properties of closure. For (P.4) suppose that A δ B and b δ C for all b in B. Then Cl(fA) ∩ Cl(fB) ≠ ∅ and Cl(fb) ∩ Cl(fC) ≠ ∅ for all b in B, which since Y is T₁, implies that fb ∈ Cl(fC) for all b in B. Thus fB ⊆ Cl(fC) or Cl(fB) ⊆ Cl(fC) so that Cl(fA) ∩ Cl(fC) ≠ ∅ showing that A δ C. For (P.6), since Cl(fA) ∩ Cl(fB) ≠ ∅, let c ∈ Cl(fA) ∩ Cl(fB) and define π to be the class of all subsets S of X such that c ∈ Cl(fS). Clearly A and B are in π and in showing that π is a cluster the demonstrations of (C.1) and (C.2) are trivial. For (C.3) suppose that Cl(fD) ∩ Cl(fC) ≠ ∅ for every C in π but that D ∈ π, i.e., c ∈ Cl(fD). Thus, c ∈ Y − Cl(fD) and since FX is regularly dense in Y there exists a subset E of X such that c ∈ Cl(fE) ⊆ Y − Cl(fD). That is, there

\(^1\) Where Cl stands for closure.
exists an \( E \) in \( \pi \) such that \( \text{Cl}(fD) \cap \text{Cl}(fE) = \emptyset \). This contradicts the hypothesis of (C.3). Thus (II) is satisfied.

For the converse suppose that (II) holds. Given \( x \) in \( X \) the class \( \pi_x \)
of all subsets \( A \) of \( X \) such that \( x \notin A \) is a cluster from \( X \), by (3.1). Thus for any subset \( A \) of \( X \), let \( \alpha \) be the set of all clusters \( \pi_a \) determined by the points \( a \) in \( A \). Let \( \bar{\alpha} \) be the set of all clusters to which \( A \) belongs. By (P.3), \( A \in \pi_a \) for each \( a \) in \( A \) and so \( \alpha \subseteq \bar{\alpha} \). We will denote \( \bar{\alpha} \), the set of all clusters from \( X \), by \( Y \).

A subset \( A \) of \( X \) absorbs a subset \( \beta \) of \( Y \) if and only if \( A \) belongs to every cluster in \( \beta \), that is, if and only if \( \bar{\alpha} \) contains \( \beta \). For any subset \( \beta \) of \( Y \) we define the closure, \( \text{cl}(\beta) \), of \( \beta \) by

(4.4) \( \pi \in \text{cl}(\beta) \) if and only if every subset \( E \) of \( X \) which absorbs \( \beta \) is in \( \pi \).

We next show that

(4.5) \( \text{cl}(\alpha) = \bar{\alpha} \).

For if \( \pi \in \text{cl}(\alpha) \) then since \( A \) absorbs \( \alpha \), \( A \in \pi \) so that \( \pi \in \bar{\alpha} \). On the other hand, if \( \pi \in \bar{\alpha} \) then \( A \in \pi \). Now let \( P \) be in every \( \pi_a \) in \( \alpha \), i.e., \( P \notin a \) for every \( a \) in \( A \) and hence \( A \subseteq P^A \). Thus, by (3.2), (2), \( P \in \pi \) so that \( \pi \in \text{cl}(\alpha) \).

We now show that the Kuratowski closure axioms are satisfied by the closure defined by (4.4).

(K.1) \( \beta \subseteq \text{cl}(\beta) \): This is trivial since if \( E \) absorbs \( \beta \) then \( E \in \pi \) for every \( \pi \in \beta \).

(K.2) \( \text{cl}(\emptyset) = \emptyset \): Suppose \( \pi \in \text{cl}(\emptyset) \). Since it is vacuously true that every subset of \( X \) absorbs \( \emptyset \), we then have that every subset of \( X \) is in \( \pi \). In particular, \( \emptyset \) and \( X \) are in \( \pi \). Thus, \( \emptyset \subseteq X \), by (C.1), contradicting (P.2).

(K.3) \( \text{cl}(\text{cl}(\beta)) \subseteq \text{cl}(\beta) \): Suppose \( \pi \in \text{cl}(\text{cl}(\beta)) \) and that \( E \) absorbs \( \beta \). By (4.4), \( E \) absorbing \( \beta \) implies that \( E \) absorbs \( \text{cl}(\beta) \). Hence \( E \in \pi \) showing that \( \pi \in \text{cl}(\beta) \).

(K.4) \( \text{cl}(\beta \cup \beta') = \text{cl}(\beta) \cup \text{cl}(\beta') \): Suppose that \( \pi \in \text{cl}(\beta \cup \beta') \) and that \( A \) absorbs \( \beta \) and \( A' \) absorbs \( \beta' \). Then, by (3.2), (1), \( A \cup A' \) absorbs \( \beta \cup \beta' \) so that \( A \cup A' \in \pi \). But, by (C.2), this means that either \( A \in \pi \) or \( A' \in \pi \), that is \( \pi \subseteq \text{cl}(\beta) \) or \( \pi \subseteq \text{cl}(\beta') \). Thus \( \pi \subseteq \text{cl}(\beta) \cup \text{cl}(\beta') \) and we have \( \text{cl}(\beta \cup \beta') \subseteq \text{cl}(\beta) \cup \text{cl}(\beta') \). On the other hand, \( \pi \in \text{cl}(\beta) \cup \text{cl}(\beta') \) implies that either \( \pi \subseteq \text{cl}(\beta) \) or \( \pi \subseteq \text{cl}(\beta') \). Now if \( E \) absorbs \( \beta \cup \beta' \), then \( E \) absorbs \( \beta \) and also absorbs \( \beta' \). Hence, \( E \in \pi \) showing that \( \pi \subseteq \text{cl}(\beta \cup \beta') \) and (K.4) holds.

To show that the topology is \( T_1 \), suppose \( \pi' \in \text{cl}(\pi) \), where \( \pi \) and \( \pi' \) are clusters from \( X \). This means that every set in \( \pi \) is also in \( \pi' \).

Thus, \( \pi \subseteq \pi' \) and by (3.2), (3), \( \pi = \pi' \). Hence, \( \text{cl}(\pi) = \pi \) for every point \( \pi \) in the space \( Y \).
Now the correspondence which assigns to each point $x$ in $X$ the cluster $\pi_x$ determined by it is a well-defined transformation mapping $X$ into $Y$ which we will denote by $f$. Note that $fA = \alpha$ for every subset $A$ of $X$, so in order to show that (4.3) holds we must show that, using (4.5),

$$A \delta B \text{ in } X \text{ if and only if } \alpha \text{ meets } \beta \text{ in } Y.$$ 

So if $A \delta B$ there exists, by (P.6), a cluster $\pi$ to which both $A$ and $B$ belong. Thus, by definition of $\alpha$, we have $\pi \in \alpha \cap \beta$. On the other hand, if $\pi \in \alpha \cap \beta$ then $A$ and $B$ are in $\pi$ so that, by (C.1), $A \delta B$.

To show that $fX = \mathfrak{X}$ is regularly dense in $Y$ suppose that $\alpha$ is an open subset of $Y$ and that $\pi \in \alpha$. We thus have $\pi \in Y - \alpha = \text{cl}(Y - \alpha)$. This means, by (4.4), that there exists some subset $E$ of $X$ such that $E$ is in every cluster of $Y - \alpha$ but that $E$ is not in $\pi$. Hence, by (C.3), there is a $C$ in $\pi$ such that $E \not\subset C$.

Since $\mathfrak{C}$ is the set of all clusters to which $C$ belongs we have $\pi \in \mathfrak{C}$. And since $E$ belongs to every cluster in $Y - \alpha$ and $E \not\subset C$, then $C$ cannot belong to any cluster in $Y - \alpha$, by (C.1). Hence $\mathfrak{C}$ is contained in $\alpha$ and we have shown that $\mathfrak{X}$ is regularly dense in $Y$.

The proof is now complete.

5. **Symmetric $P_1$-spaces.** A $P_1$-space $(X, \delta)$ in which $\delta$ satisfies the additional axiom

$$x \delta y \text{ implies } x = y \text{ for all points } x, y \in X$$

is called a *symmetric $P_1$-space* (see [3]). The following theorem follows directly from (C.1) and (5.1).

(5.2) **Theorem.** Every cluster $\pi$ from a symmetric $P_1$-space $(X, \delta)$ possesses at most one point.

(5.3) **Theorem.** Given a set $X$ and some binary relation $\delta$ on the power set of $X$, the following are equivalent:

(i') There exists a $T_1$ topological space $Y$ in which $X$ can be topologically imbedded as a regularly dense subset so that (1.1) holds.

(ii') $\delta$ is a symmetric $P_1$-relation satisfying (P.6).

**Proof.** The demonstration is similar to that of Theorem (4.2). To see that (5.1) holds, note that $\mathfrak{x} \cap \mathfrak{y} \not= \emptyset$ implies that $x \cap y \not= \emptyset$, or $x = y$.

To show that our imbedding is topological we note first that, because of (5.2) the correspondence between $X$ and $\mathfrak{X}$ induced by the identification of $x$ with the cluster $\pi_x$ determined by it is one-to-one. To see that the mapping is bicontinuous we must show that if $A$ is a
subset of $X$, $x \in A^\delta$ if and only if $\pi_x \in \text{kl}(\alpha)$, where $\text{kl}(\alpha)$ is the closure of $\alpha$ in $\mathfrak{K}$ relative to the space $Y$.

So suppose $x \in A^\delta$ and that $P$ absorbs $\alpha$. Then $P$ is a member of every $\pi_a$ in $\alpha$ and it follows that $a \in P$ for every $a$ in $A$. Thus, $A \subseteq P^\delta$ and since $A \subseteq \pi_x$ we have, from (3.2), (2), that $P \subseteq \pi_x$. Thus, $\pi_x \subseteq \text{kl}(\alpha)$.

On the other hand, suppose $\pi_x \subseteq \text{kl}(\alpha)$. Then since $A$ absorbs $\alpha$ we have $A \subseteq \pi_x$, i.e., $A \in x$ and hence $x \in A^\delta$. This completes the proof.

References


Laboratory for Electronics, Inc., Monterey, California