EXTREME HAMILTONIAN CIRCUITS. RESOLUTION OF THE CONVEX-ODD CASE

FRED SUPNICK AND LOUIS V. QUINTAS

Let \( n \) points in the Euclidean plane fall on the boundary \( B \) of their convex hull. It is well known that a shortest polygon passing through these points coincides with \( B \). But, it is not known how to explicitly indicate a longest polygon having these \( n \) points as vertices. In this paper we do this for the case where \( n \) is odd.\(^1\)

**Theorem.** Let

\[
(1) \quad P_1, P_3, P_5, \ldots, P_{2p-1}, P_2, P_4, \ldots, P_{2p-2}
\]

be points in the plane which fall on the boundary \( B \) of their convex hull in the stated (linear or cyclic) order (accordingly as the points (1) are or are not collinear). Then \([P_1P_3 \cdots P_{2p-1}]\)\(^2\) is a longest polygon with (1) as vertices; if no three points of (1) are collinear, it is the only one.

**Proof.** Case I. Suppose no three points of (1) are collinear. An edge of a polygon intersected by all noncontiguous edges must have the vertices of the polygon alternatively on either side, and is thus an edge of \([P_1P_3 \cdots P_{2p-1}]\). This implies that \([P_1P_2 \cdots P_{2p-1}]\) is the only polygon with each closed edge intersecting every other closed edge.

The symbol \([V_1 \cdots V_{i-1}(V_i \cdots V_j)V_{j+1} \cdots V_n]\) will denote the polygon \([V_1 \cdots V_{i-1}V_jV_{j-1} \cdots V_iV_{j+1} \cdots V_n]\); the operation \([\cdots (\cdots) \cdots]\) will be referred to as an **arcinversion** (cf. [1, p. 180]). Let \(h=[R_1 \cdots R_{2p-1}]\) denote any polygon having (1) as vertices and which is distinct from \([P_1 \cdots P_{2p-1}]\). We show that there is an arcinversion which yields a longer polygon. Let \(i\) denote the smallest integer such that the closed edge \(R_iR_{i+1}\) of \(h\) does not intersect at least one of the closed edges \(R_iR_2, R_2R_3, \ldots, R_{i-1}R_i\) of \(h\); \(i\) of course satisfies \(2 < i < 2p - 1\). Then, the vertices \(R_{i-1}\) and \(R_i\) define the following partition of \(B\): \(B_1 \cup B_2 \cup \{R_{i-1}, R_i\}\), where \(B_1\) is the component of \(B - \{R_{i-1}, R_i\}\) which contains \(R_i\).

**Case A.** \(R_i-2\) and \(R_{i+1}\) in the same component of \(B - \{R_{i-1}, R_i\}\).

Presented to the Society, February 23, 1963; received by the editors February 11, 1963.

\(^1\) *Added in proof.* The remaining case, where \(n\) is even, has recently been resolved by the authors (Abstract 611-60, Notices Amer. Math. Soc. 11 (1964), 335).

\(^2\) The symbols for polygons are to be considered cyclic and symmetric.
(i) Suppose \( i \) is odd. Then \( R_{i+1} \) is in \( \mathcal{C}_2 \) and the closed edge \( R_i R_{i+1} \) does not intersect the closed edge \( R_i R_2 \). For, if \( R_i R_{i+1} \cap R_i R_2 \neq \emptyset \) (\( \emptyset \) denotes the empty set), then the closed edge \( R_i R_{i+1} \) would intersect each of the closed edges \( R_2 R_3, R_3 R_4, \ldots, R_{i-1} R_i \). The arcinversion \( [R_1(R_2 \cdots R_i)R_{i+1} \cdots R_{2p-1}] \) yields a polygon which is longer than \( h \).

(ii) Suppose \( i \) is even. Then \( R_{i+1} \) is in \( \mathcal{C}_2 \) and the closed edge \( R_i R_{i+1} \) does not intersect the closed edge \( R_2 R_3 \). Thus, the arcinversion \( [R_1 R_2(R_3 \cdots R_i)R_{i+1} \cdots R_{2p-1}] \) yields a polygon which is longer than \( h \).

Case B. \( R_{i-2} \) and \( R_{i+1} \) in different components of \( B - \{R_{i-1}, R_i\} \). Let \( \mathcal{C}_1 \) denote the component \( B_1 \) or \( B_2 \) which contains at most \( p - 2 \) vertices, and \( \mathcal{C}_3 \) the component which has at least \( p - 1 \) vertices. Let the vertices of \( h \) be renumbered consecutively as follows: \( h = [S_1 S_2 \cdots S_{2p-1}] \) with \( R_{i-1} R_i = S_i S_2 \) or \( S_2 S_1 \) so that \( S_3 \) is in \( \mathcal{C}_1 \).

Let \( k \) denote the number of vertices in \( \mathcal{C}_1 \). We first show that there is at least one edge of \( h \) which has both vertices in \( \mathcal{C}_2 \). There are at most \( 2k \) edges incident to the vertices in \( B \setminus \{S_1, S_2\} \) which terminate at vertices in \( \mathcal{C}_2 \). There are \((2p-1)-(k+2)\) vertices in \( \mathcal{C}_2 \). Thus, there are at least the following number of edges of \( h \) which have both vertices in \( \mathcal{C}_2 \)

\[
N = \frac{2((2p - 1) - (k + 2)) - 2k}{2} = 2p - 2k - 3.
\]

Since \( k \leq p - 2 \), we have \( N \geq 2p - 2(p - 2) - 3 = 1 \).

Let \( S_i S_{i+1} \) denote an edge of \( h \) which has both vertices in \( \mathcal{C}_2 \). Then, either \( S_i S_i \cap S_{i+1} \neq \emptyset \) or \( S_i S_{i+1} \neq \emptyset \). In the former case the arcinversion \( [S_1(S_2 \cdots S_i)S_{i+1} \cdots S_{2p-1}] \) yields a polygon which is longer than \( h \), and in the latter case the arcinversion

\[
[S_1 S_2(S_3 \cdots S_i)S_{i+1} \cdots S_{2p+1}]
\]

yields a polygon which is longer than \( h \).

Remark. We note that points (1) satisfying Case I have the property that the longest (also, shortest) polygon can be obtained from any other polygon by a sequence of arcinversions each of which strictly increases (decreases) the length of the polygon to which it is applied (cf. [1, Remark III, p. 181]).

Case II. Suppose \( B \) has support lines passing through at least three points of (1). If the points of (1) are not all collinear, let \( P \) be a point in the interior of the convex hull of (1) and \( B(t) \) (0 \( \leq t < 1 \)) a family of strongly convex curves circumscribing \( B \) and converging to \( B \) as \( t \) approaches 1. If the points of (1) are all collinear, let \( P \) be a point in

License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use
one of the open half-planes defined by the line on which the points (1) lie and \( B(t) \) \((0 \leq t < 1)\) a family of strongly convex arcs having \( P_1 \) and \( P_{2p-2} \) as endpoints, converging to \( B \) as \( t \) approaches 1, and lying in the closed half-plane which does not contain \( P \). Let \( P_i(t) \) be the intersection of \( B(t) \) with the ray emanating from \( P \) and passing through \( P_i \) \((1 \leq i \leq 2p-1)\). Then, for each \( t \) \((0 \leq t < 1)\), Case I implies \([P_1(t) \cdots P_{2p-1}(t)]\) is longer than any other polygon \([P_{i_1}(t) \cdots P_{i_{2p-1}}(t)]\). Thus, \([P_1 \cdots P_{2p-1}]\) is a polygon of maximum length.

**Remark.** We note that in Case II \([P_1 \cdots P_{2p-1}]\) is not necessarily the only longest polygon. For example, in a set (1) for which \( P_1, P_2, P_3, \cdots, P_{2p-1}, P_2, P_4 \) are collinear the polygons \([P_1 \cdots P_{2p-1}]\) and \([P_1(P_2P_3P_4)P_5 \cdots P_{2p-1}]\) have the same length.

**Reference**


City College, New York and

St. John’s University, Jamaica, New York