A THEOREM ON MAXIMUM MODULUS

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Introduction. If $D$ is a domain in the plane of complex numbers, then every analytic function achieves its maximum modulus only at the boundary. We phrase this by asserting that if $f$ is analytic in $D$, and $f$ is not constant, and if $x \in \partial D$ has the property that
\[
\lim \sup_{z \to x} |f(z)| = \sup_{D} |f(z)|,
\]
then $x \in \partial(D)$, the boundary of $D$. If $x$ is the only point for which the above identity holds, then $x$ is the peak point for $f$ in $D$. If $x$ is the peak point for a bounded analytic function in $D$, then $x$ is a peak point of $D$. We are interested in knowing which boundary points of $D$ can be peak points.

W. Rudin [1] defines a boundary point $x$ of a domain $D$ as a removable boundary point if every function bounded and analytic in $D$ can be continued at $x$. All boundary points which are not removable are essential.

We shall show that a point is a peak point of $D$ if and only if it is an essential boundary point.

Lemma 1. If $D$ is a simply connected domain and $x \in \partial(D)$ is linearly accessible from the interior, then there is a schlicht mapping $f$ of $D$ into the open unit disc such that $\lim_{z \to x} f(z) = 1$ and $\lim \sup_{z \to x} |f(z)| < 1$ if $\xi \in \partial(D)$, $\xi \neq x$.

Proof. Let $(x, x+\alpha]$ be a line segment contained in $D$. Then the function
\[
f(z) = g^{-1}(2 + 4(\alpha^{-1}(z-x))^2),
\]
where $g(z) = z + 1/z$, $|z| < 1$, is a schlicht function with the properties:
1. $\lim_{z \to x} f(z) = 1$,
2. $|x-y| > \epsilon$, $y \in D \Rightarrow 1 - |f(y)| > \eta = \eta(\epsilon, D) > 0$,
as we shall show. Let $x_i \to x$, $x_i \in D$ and $|y-x| > \epsilon$. If we examine the mapping $z \to f_1(z) = \alpha^{-1}(z-x)$, it is a linear mapping taking $[x, x+\alpha]$ onto $[0, 1]$. $\alpha^{-1}(x_i-x) \to 0$, $|f_1(y) - 0| > \epsilon/|\alpha|$. Since $D$, and thus $f_1(D)$, is simply connected, we can apply a square root, with $(+1)^{1/2} = +1$, and get $f_2(z) = (f_1(z))^{1/2}$, $(\alpha^{-1}(x_i-x))^{1/2} \to 0$, and

Received by the editors February 13, 1963.

1 This research was supported by the University of Wisconsin under contract No. AF 49(638)-868 with the Air Force Office of Scientific Research.
We note that each point of \((0, 1]\) is an interior point of \(f_2(D)\), so that each point of \([-1, 0)\) is an exterior point of \(f_2(D)\). Thus, \(\overline{f_2(D)}\) meets \([-1, 0)\) only at 0. Set \(\eta_1 = \eta_1(\epsilon, D) = d([-1, 0], f_2(D) - N(0, \epsilon_1)) > 0\). Then \(d(f_2(y), [-1, 0]) \geq \eta_1\). Setting \(f_3(z) = 2 + 4(f_2(z))\), we have \(f_3(x_i) \to 2\) and \(d(f_3(y), [-2, 2]) \geq 4\eta_1\). We now observe that for our function \(g\), \(g(z) \to 2\) iff \(z \to 1\) and \(|z| \leq 1 - \epsilon \Rightarrow d(g(z), [-2, 2]) > \epsilon^2\). Thus \(f(y) = g^{-1}(f_3(x_i)) = g^{-1}(2 + 4(\alpha^{-1}(x_i - z))^{1/2}) \to 1\) and \(|f(y)| = |g^{-1}(f_3(y))| < 1 - (4\eta_1)^{1/2}\) so that \(1 - |f(y)| > (4\eta_1)^{1/2} = \eta(\epsilon, D) > 0\), as promised.

**Lemma 2.** The set of peak points of any domain is closed.

**Proof.** Let \(a_i\) be the peak point of \(f_i\) in \(D\), \(i = 1, 2, \ldots\), and let \(a_i \to a\). We assume that \(\sup_{z \in D} |f_i| = 1\). Let \(N_i\) be a neighborhood of \(a_i\), \(i = 1, 2, \ldots\), with \(\text{diam}(N_i) \to 0\) and pairwise disjoint. Since \(a_i\) is a peak point of \(f_i\), \(f_i\) is bounded away from 1 off \(N_i\), so that for an appropriate integer \(m_i\), we have

\[
\sup_{z \in D - N_i} |f_i^{m_i}(z)| < 4^{-i}.
\]

Set \(g_i(z) = f_i^{m_i}(z)\). We will now generate (inductively) a sequence \(\{b_i\}\) such that

1. \(\left| b_i \right| \leq 2\),
2. \(\sum_{i=1}^{n} b_i g_i\) is bounded and analytic in \(D\),
3. \(\sup_{z \in N_i} |\sum_{i=1}^{n} b_i g_i| = 2 - 1/n\).

We take \(b_1 = 1\). Given \(b_1, \ldots, b_{n-1}\), we find \(b_n\) as follows:

Set \(h_n(\xi) = \sup_{z \in N_n} |b_1 g_1 + \cdots + b_{n-1} g_{n-1} + \xi : g_n|, |\xi| \leq 2\). Then \(h_n(0) = \sup_{z \in N_n} |b_1 g_1 + \cdots + b_{n-1} g_{n-1}| < \sum_{i=1}^{n-1} 2 \cdot 4^{-i} < 2/3\). We now choose a point \(y_n\) such that \(g_n(y_n) > 1 - (1/2n)\), and set

\[
c_n = 2 \frac{b_1 g_1(y_n) + \cdots + b_{n-1} g_{n-1}(y_n)}{g_n(y_n)} \frac{g_n(y_n)}{b_1 g_1(y_n) + \cdots + b_{n-1} g_{n-1}(y_n)}.
\]

Then

\[
h_n(c_n) \geq \left| b_1 g_1(y_n) + \cdots + b_{n-1} g_{n-1}(y_n) + c_n g_n(y_n) \right| \geq \left| c_n g_n(y_n) \right| > 2 - \frac{1}{n},
\]

*This is because the image under \(g^{-1}\) of the circle \(x^2 + y^2 = \epsilon^2\) is the ellipse \((u^2/a^2) + (v^2/b^2) = 1\), where \(a = c + 1/c, b = c - 1/c\). If \(\epsilon = 1 - \epsilon\) then the image of \(x^2 + y^2 < (1 - \epsilon)^2\) is the exterior of the indicated ellipse. The closest approach of this set to the line \([-2, 2]\) is at the vertex, where the distance is \(\epsilon^2/(1 - \epsilon)\).*
since \( \sum_{i=1}^{n-1} b_i g_i(z) \) and \( c_n g_n(y_n) \) have the same argument. Since
\( h_n(0) < 2/3 < 2 - 1/n < h_n(c_n) \), there is a point \( b_n \) in the disc \( |z| \leq 2 \) such that \( h_n(b_n) = 2 - 1/n \). For \( z \in N_n \), we then have
\[
\left| \sum_{i=1}^{n-1} b_i g_i(z) \right| \leq \left| \sum_{i=1}^{n} b_i g_i(z) \right| + \sum_{i=n+1}^{\infty} |b_i| \cdot g_i(z) < 2 - \frac{1}{n} + \sum_{i=n+1}^{\infty} 2 \cdot 4^{-i} = 2 - \frac{1}{n} + \frac{2}{3 \cdot 4^n}.
\]
Similarly,
\[
\sup_{z \in N_n} \left| \sum_{i=1}^{n} b_i g_i(z) \right| > 2 - \frac{1}{n} - \frac{2}{3 \cdot 4^n}.
\]
For \( z \in D - \bigcup_{i=1}^\infty N_i \), we have
\[
\left| \sum_{i=1}^{\infty} b_i g_i(z) \right| < \sum_{i=1}^{\infty} 2 \cdot 4^{-i} = 2/3.
\]
Thus \( \left| \sum_{i=1}^{\infty} b_i g_i(z) \right| < 2 \) for all \( z \in D \). If \( K \) is a compact subset of \( D \), then \( K \) meets only finitely many \( N_i \), so that the series converges uniformly and absolutely on \( K \). Thus \( f(z) = \sum_{i=1}^{\infty} b_i g_i(z) \) is analytic. Finally, it is clear that \( |f(z)| \) is close to 2 only inside \( N_i \), for large \( i \), that is, only around \( a \).

**Lemma 3.** If \( D \) is any domain, the boundary points linearly accessible from the interior are dense in \( \delta(D) \).

**Proof.** Let \( x \in \delta(D), \ eps > 0 \). Let \( y \in D, |y - x| < \eps \). Let \( x_1 \) be the point of \( \delta(D) \cap [x, y] \) lying nearest to \( y \). Then \( |x - x_1| < \eps \) and \( x_1 \) is linearly accessible.

**Theorem 1.** If \( D \) is simply connected, and its boundary consists of more than one point, then every boundary point is a peak point.

**Proof.** Direct consequence of Lemmas 1, 2, 3.

**Lemma 4.** If \( x \in \delta(D) \) and the component of \( x \) in\( D', K(x) \neq \{x\}, \) then \( x \) is a peak point of \( D \).

**Proof.** Let \( y \in K(x), y \neq x \). Let \( K_1 \) be a sub-continuum of \( K(x) \), containing \( y \) but not containing \( x \). There is a conformal mapping \( \phi \) of \( K_1' \) onto the open unit disc, and \( \phi \) is a homeomorphism around \( x \). Then \( \phi(K(x))' \) is a simply connected domain, and each linearly accessible point of \( \delta(\phi(K(x))') \) is a peak point of the kind described in Lemma 1. If we now restrict our functions to \( \phi(D) \), each of these points is still a peak point, though the point may no longer be linearly accessible. Now, by Lemma 2, \( x \) is a peak point.
Lemma 5. If $K(D)$ denotes the closure of the union of those components of $\delta(D)$ which are not single points, then every point in $K(D)$ is a peak point of $D$.

Proof. Clear from Lemmas 4 and 2.

Definition. A set $S$ is a Painlevé null set (called a $p$-null set) if the algebra of bounded analytic functions on $S'$ consists of the constants alone.

Definition. A point $x \in \delta(D)$ is called a $p$-essential boundary point if for each $\epsilon > 0$, $N(x, \epsilon) \cap \delta(D)$ is not a $p$-null set.

Lemma 6. Let $x \in \delta(D)$ and $x \in K(D)$. Then if $x$ is a $p$-essential boundary point, $x$ is a peak point of $D$.

Proof. Let $N$ be a neighborhood of $x$ in which $\delta(D)$ is totally disconnected. Since the $p$-essential boundary points form a perfect set, let $x_i \to x$ be a sequence of $p$-essential boundary points. Let $N_i$ be a sequence of neighborhoods with $N_i \subset N$, $x_i \in N_i$ no two $N_i$ intersecting, and $\text{diam}(N_i) \to 0$. Let $M_i$ be open with $x_i \in M_i \subset \overline{M_i} \subset N_i$ for each $i$. Then $K_i = M_i \cap \delta(D)$ is not a $p$-null set and thus we can find $f_i$ such that $f_i$ is analytic on $K_i$ (including $\infty$) and $f_i$ is not constant. We can assume that $\sup_{x \in K_i} |f_i| = 1$. Then $\sup_{x \in K_i} |f_i| < 1$, so we can choose a sequence of integers $m_i$ so that

$$\sup_{N_i} |f_i^{m_i}| < 4^{-i}.$$ 

Set $g_i(z) = f_i^{m_i}(z)$. Since $D'$ is nowhere dense in $N_i$, $\sup_{D'} |g_i| = 1$, and $\sup_{D-N_i} |g_i| < 4^{-i}$.

Using the same technique as in Lemma 2, we choose $b_n$ so that

1. $|b_n| \leq 2$,
2. $\sup_{N_i} \left| \sum_{i=1}^n b_i g_i \right| = 2 - 1/n$,
3. $f(z) = \sum_{i=1}^n b_i g_i(z)$ is bounded and analytic in $D$.

Furthermore, we deduce that

$$2 - \frac{1}{n} - \frac{2}{3} \cdot 4^{-n} < \sup_{N_n} |f| < 2 - \frac{1}{n} + \frac{2}{3} \cdot 4^{-n},$$

and $\sup_{D-U_i N_i} |f| < 2/3$. Thus, $f$ has a peak at $x$.

Theorem 2. If $x$ is a $p$-essential boundary point, then $x$ is a peak point.

Proof. If $x \in K(D)$, this is true by Lemma 5; otherwise by Lemma 6.

Theorem 3. $x$ is a peak point iff $x$ is an essential boundary point.
Proof. By a theorem of W. Rudin [1], $x$ is an essential boundary point iff $x$ is a $p$-essential boundary point. From the remarks above, every $x \in K(D)$ is a peak point of a function which has no limit at $x$. Thus, no point of $K(D)$ is removable. If $x \in \delta(D) - K(D)$ and $x$ is removable, then each $f$ is continuably there. Since $\delta(D)$ is nowhere dense around $x$, this extension does not change the maximum modulus nor the lim sup at $x$. By the maximum modulus theorem, $f(x)$ is not the maximum. By continuity, $x$ is not a peak for any $f$.

It would be interesting to know when $x$ is the peak point of a bounded analytic function $f$ which has a limit at $x$, or whose modulus has a limit at $x$. It is clear from previous remarks that every essential boundary point is the peak point of a function which does not have these limits. This question is open.

Reference