GROUPS OF EXPONENT 8 SATISFY THE 14TH ENGEL CONGRUENCE[1]

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I. **Introduction and the Main Theorem.** Let G be a group and n a positive integer. We shall say that G satisfies the nth Engel congruence if

$$(y, x; n) \equiv 1 \mod G_{n+2} \qquad \text{for all } x, y \in G.$$

As usual, $(y, x; k)$ is defined inductively by

$$(y, x; 1) = (y, x) = y^{-1}x^{-1}yx,$$
$$(y, x; k) = ((y, x; k-1), x) \qquad \text{for } k > 1;$$

and G_m denotes the mth term of the lower central series for G.

Kostrikin [2] has made use of Engel congruences in his solution of the Restricted Burnside Problem for prime exponent. It is hoped that the following theorem will be of use in attacks on the Restricted Burnside Problem for exponent 8.

MAIN THEOREM. *Let G be a group of exponent 8 and let $x \in G$. Let t be a positive integer $\geqq 3$ and let $y_t \in G_t$. Let*

$$m = \min(t + 13, 2t + 10, 3t + 7, 4t + 4, 5t + 1).$$

Then

$$(y_t, x; 4)^4 \equiv 1 \mod G_m, \tag{1}$$

$$(y_t, x; 8)^2 \equiv 1 \mod G_m, \tag{2}$$

$$(y_t, x; 12) \equiv 1 \mod G_m. \tag{3}$$

REMARK. When t is replaced by 3 and y_3 is replaced by (y, x, x), where $y \in G$, (3) reduces to the 14th Engel congruence.

Throughout this paper we shall be investigating the canonical expression, in terms of basic commutators, for the 8th power of a product of two group elements [1, pp. 179–182]; e.g.,

$$(xy)^8 = x^8y^8(y, x)^{28} \cdots . \tag{4}$$

We shall use formulas we derived in an earlier paper [3] to compute the exponents of various commutators appearing in (4), but it would

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[1] This result appears in Chapter 3 of the author's doctoral thesis written at the University of Wisconsin under the direction of Professor R. H. Bruck.

not be impossible for the reader to compute these exponents directly by "collecting" $(xy)^8$ in the usual manner [1, p. 165].

II. Lemmas.

LEMMA 1. *Let a, b be positive integers and G a group of exponent* 8. *Let $y_a \in G_a$ and $z_b \in G_b$. Then*

$$(y_a, z_b)^4 \equiv 1 \bmod G_q,$$

where $q = \min(3a+b, a+3b)$.

PROOF. For notational convenience we write y_a as y and z_b as z, and we understand that all congruences are mod G_q.

$$1 \equiv (zy)^8 \equiv (y, z)^4 (y, z, y)^4$$

yields

$$(y, z)^4 \equiv (y, z, y)^4$$

which yields

$$((y, z)^4, y) \equiv 1.$$

But

$$((y, z)^4, y) \equiv (y, z, y)^4.$$

LEMMA 2. *Let G be a group of exponent* 8. *Let t be a positive integer and let $y_t \in G_t$. If $t \geqq 3$, then*

$$(y_t, x; 4)^2 \equiv 1 \bmod G_{t+8}. \tag{5}$$

PROOF. For notational convenience we write y_t as y and understand all congruences to be mod G_{t+8}. Since $t+8 = \min(t+8, 2t+5, 3t+2, 4t)$ for $t \geqq 3$, the only commutators we have to consider are

$$x, y, (y, x), (y, x, x), (y, x, y), (y, x; 3), (y, x, x, y), (y, x, y, y), (y, x; 4),$$
$$((y, x; 3), y), (y, x, x; y, x), (y, x; 5), ((y, x; 4), y), (y, x, x, x; y, x),$$
$$(y, x; 6), \text{ and } (y, x; 7)$$

where, for example, $(y, x, x; y, x)$ stands for $((y, x, x), (y, x))$. In order and modulo 8, the exponents of these commutators in the collected form of $(xy)^8$ are (see (4) and the remark following) 0, 0, 4, 0, 4, 6, 2, 2, 0, 2, 2, 4, 4, 2, 0, and 1. Hence, using Lemma 1, we obtain

$$\begin{aligned} 1 \equiv (xy)^8 \equiv (y, x)^4 (y, x; 3)^6 (y, x, x, y)^2 (y, x, y, y)^2 (y, x, x, x, y)^2 \\ \cdot (y, x, x; y, x)^2 (y, x; 5)^4 (y, x, x, x; y, x)^2 (y, x; 7) \end{aligned} \tag{6}$$

where the nine commutators are understood to be multiplied in order.

Next we replace each occurrence of x in (6) by x^2 to obtain

$$1 \equiv (y, x; 2)^4(y, x; 5)^4(y, x; 6)^6(y, x, x, x, x, y)^2. \tag{7}$$

For example, one easily verifies that in collected form

$$(y, x^2) = (y, x)^2(y, x, x).$$

Then one gets a collected expression for (y, x^2, x^2) by writing out

$$\begin{aligned}(y, x^2, x^2) &= (y, x^2)^{-1}x^{-2}(y, x^2)x^2 \\ &= (y, x, x)^{-1}(y, x)^{-2}x^{-2}(y, x)^2(y, x, x)x^2,\end{aligned}$$

collecting the factor, $(y, x)^2(y, x, x)x^2$, and performing all possible cancellations. Similarly one gets a collected expression for $(y, x^2; 3)$ by writing out

$$(y, x^2; 3) = (y, x^2; 2)^{-1}x^{-2}(y, x^2; 2)x^2,$$

substituting in the collected expression for $(y, x^2; 2)$, and collecting the factors appearing to the right of x^{-2}. The same plan is used to get a collected expression for each of the commutators appearing in (6) (with x replaced by x^2).

Replacing y by (y, x) in (6) (note that $y \in G_i$ implies $(y, x) \in G_i$), we obtain

$$1 \equiv (y, x; 2)^4(y, x; 4)^6(y, x, x, x; y, x)^2. \tag{8}$$

Comparing (7) and (8) we see that

$$(y, x; 4)^6 \equiv (y, x; 5)^4(y, x, x, x, x, y)^2(y\, x, x, x; y, x)^6(y, x; 6)^6. \tag{9}$$

But we can replace y by $(y, x; 3)$ in (7) to obtain $(y, x; 5)^4 \equiv 1$, and we can replace y by $(y, x; 2)$ in (9) to obtain $(y, x; 6)^6 \equiv 1$. Hence (9) becomes

$$(y, x; 4)^6 \equiv (y, x, x, x, x, y)^2(y, x, x, x; y, x)^6. \tag{10}$$

Commuting both sides with y and proceeding as in the proof of Lemma 1, we reduce (10) to

$$(y, x; 4)^6 \equiv (y, x, x, x; y, x)^6. \tag{11}$$

Next we solve (6) for $(y, x; 3)^2$ and commute both sides of the resulting congruence with (y, x) to obtain

$$((y, x; 3)^2, (y, x)) \equiv 1. \tag{12}$$

But

$$((y, x; 3)^2, (y, x)) \equiv (y, x, x, x; y, x)^2,$$

and hence (11) yields (5).

COROLLARY 1. *Let G be a group of exponent 8, r and s be positive integers with $r \geqq 3$, and $y_r \in G_r$, $z_s \in G_s$. Then*

$$((y_r, x; 4), z_s)^2 \equiv 1 \mod G_{r+s+8} \tag{13}$$

and hence

$$(z_s, (y_r, x; 4))^2 \equiv 1 \mod G_{r+s+8}. \tag{14}$$

PROOF. Use the well-known commutator identity,

$$(ab, c) = (a, c)(a, c, b)(b, c).$$

III. **Proof of the Main Theorem.** Let G, m, x, t, and y_t be as in the Main Theorem, but, for convenience of notation, write y_t as y and understand all congruences to be mod G_m. Form and order (within a weight class) basic commutators from x and y according to the rule:

$$\begin{aligned} &x < y, \qquad (v_1, v_2) < (u_1, u_2) \text{ if} \\ &\text{(a) weight of } v_1 > \text{weight of } u_1 \text{ or} \\ &\text{(b) weight of } v_1 = \text{weight of } u_1 \text{ and } v_1 < u_1 \text{ or} \\ &\text{(c) } v_1 = u_1 \text{ and } v_2 < u_2. \end{aligned} \tag{15}$$

The fundamental relation we shall investigate is

$$(xy)^8 \equiv \prod u^{e(u)} \mod G_m, \tag{16}$$

where the product is ordered as u ranges through the ordered list of basic commutators given by (15).

The first step in our proof will be to replace each occurrence of x in (16) by x^4. Rather than first computing $e(u)$ for every u appearing in (16), a monumental task, and then replacing each x by x^4, we shall instead get an expression, in terms of commutators in x and y, for the commutator u_4 which u becomes when x is replaced by x^4. For many u's we shall observe that $u_4 \in G_m$. Thus we shall need to compute $e(u)$ for only those u's, appearing in (16), for which $u_4 \notin G_m$—an easy bit of computation.

Lemma 1 and Corollary 1 are used repeatedly in computing the following congruences.

$$\begin{aligned} (y, x^4) \equiv\ & (y, x)^4 (y, x, x)^6 (y, x; 3)^4 (y, x; 4)(y, x, x, x; y, x) \\ & \cdot (y, x, x; y, x; y, x)(y, x, x; y, x; y, x, x)(y, x, x, x; y, x; y, x, x)^3. \end{aligned} \tag{17}$$

NOTE. (17) is obtained by the same sort of technique as was used in going from (6) to (7); namely, write

$$(y, x^4) = y^{-1}x^{-4}yx^4,$$

collect the right "half," yx^4, and perform all possible cancellations. Use Lemma 1 and Corollary 1 to simplify.

$$\begin{aligned}(y, x^4, x^4) \equiv\ & (y, x; 4)^4(y, x; 6)^4(y, x; 8)((y, x; 6), (y, x, x))^7 \\ & \cdot(y, x, x, x; y, x; x)^4(y, x, x, x; y, x; x; x)^6 \\ & \cdot(y, x, x, x; y, x; x; x; x; x)c_{2t+9}d_{3t+6},\end{aligned} \tag{18}$$

where $c_{2t+9} \in G_{2t+9}$ and $d_{3t+6} \in G_{3t+6}$.

NOTE. (18) is obtained by writing

$$(y, x^4, x^4) = (y, x^4)^{-1}x^{-4}(y, x^4)x^4,$$

replacing (y, x^4), in the above, by its expression in (17), collecting all factors appearing to the right of x^{-4} in the resulting expression, and performing cancellations. Lemma 1 and Corollary 1 are again used to simplify. (19) through (24) are obtained similarly.

$$(y, x^4; 3) \equiv (y, x; 8)^4(y, x; 10)^4(y, x; 12). \tag{19}$$

$$(y, x^4, y) \equiv (y, x, y)^4(y, x, y; y, x)^6(y, x, x, y)^6(y, x, x, x, x, y)f_{3t+4}, \tag{20}$$

where $f_{3t+4} \in G_{3t+4}$.

$$(y, x^4, x^4, y) \equiv ((y, x; 8), y)(y, x, x, x; y, x; x; x; y)^6. \tag{21}$$

$$(y, x^4, y, y) \equiv (y, x, y; y, x; y)^6(y, x, x, y, y)^6((y, x; 4), y, y). \tag{22}$$

$$(y, x^4, y, y, y) \equiv (y, x, x, y, y, y)^6. \tag{23}$$

$$u_4 \equiv 1 \qquad \text{for all other } u \text{ appearing in (16).} \tag{24}$$

Thus the only commutators appearing in (16) which are not necessarily sent into G_m when x is replaced by x^4 are:

$$\begin{aligned}& x, y, (y, x), (y, x, x), (y, x, y), (y, x; 3), (y, x, x, y), \\ & (y, x, y, y), \text{ and } (y, x, y, y, y).\end{aligned} \tag{25}$$

The exponents, mod 8, corresponding to this list of basic commutators are 0, 0, 4, 0, 4, 6, 2, 2, and 2 so that (16), with x replaced by x^4, is

$$\begin{aligned}1 \equiv (x^4y)^8 \equiv\ & (y, x^4)^4(y, x^4, y)^4(y, x^4; 3)^6(y, x^4, x^4, y)^2 \\ & \cdot(y, x^4, y, y)^2(y, x^4, y, y, y)^2,\end{aligned} \tag{26}$$

and, substituting from (17), (20), (19), (21), (22), and (23) into (26), we obtain

$$1 \equiv (y, x; 4)^4$$

which is (1).

The next step in our proof will be to replace y by (y, x) in (16) and then to square the resulting congruence. Again it will not be necessary for us to compute very many of the $e(u)$ because, first of all, many of the u's are sent into G_m when y is replaced by (y, x); and, secondly, we can apply Corollary 1 to any commutator of form (13) or (14) as long as its exponent is even—it need not be 2.

Replacing y by (y, x) in (16) and using Lemma 1, we obtain

$$\begin{aligned} 1 \equiv\ & (y, x, x)^4(y, x; 4)^6(y, x, x, x; y, x)^2(y, x, x; y, x; y, x)^2 \\ & \cdot((y, x; 4), (y, x))^2(y, x, x, x; y, x, x)^2(y, x; 6)^4 \\ & \cdot((y, x; 5), (y, x))^*(y, x, x, x, x; y, x; y, x)^* \\ & \cdot((y, x; 4), (y, x, x))^*(y, x, x; y, x; y, x, x, x)^6 \\ & \cdot((y, x; 6), (y, x))^*((y, x; 5), (y, x, x))^*((y, x; 4), (y, x; 3))^* \\ & \cdot(y, x, 8)((y, x; 7), (y, x))^*((y, x; 6),(y, x, x))^* \\ & \cdot((y, x; 5), (y, x; 3))^*((y, x; 8), (y, x))^*((y, x; 7)(y, x, x))^* \\ & \cdot((y, x; 6), (y, x; 3))^*((y, x; 5), (y, x; 4))^* \end{aligned} \tag{27}$$

where * denotes an exponent that we have not bothered to compute.

Squaring (27) and using Lemma 1 and Corollary 1—note that 2 * is an even number—we obtain

$$1 \equiv (y, x; 4)^4(y, x; 8)^2$$

which, in view of (1), reduces to

$$1 \equiv (y, x; 8)^2$$

which is (2).

Finally, to prove (3), we replace y by $(y, x; 5)$ in (16) to obtain

$$1 \equiv (y, x; 6)^4(y, x; 8)^6(y, x; 10)^4(y, x; 12)$$

which, in view of (1) and (2), reduces to (3).

References

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