-FRAMES IN EUCLIDEAN \( k \)-SPACE

J. C. CANTRELL

1. Introduction. An \( n \)-frame \( F_n \) is a union of \( n \) arcs, \( F_n = \bigcup_{i=1}^{n} A_i \), with a distinguished point \( p \) such that, if \( n = 1 \), \( p \) is an end point of \( A_1 \), and if \( n > 1 \), \( p \) is an end point of each \( A_i \) and \( A_i \cap A_j = p, \ i \neq j \).

We introduce the distinguished point in order to differentiate between a 1-frame and a 2-frame. A 1-frame is an arc with an end point distinguished and a 2-frame is an arc with an interior point distinguished. This difference will keep certain logical difficulties from arising in the inductive proof of Theorem 1.

In \( E_k \) let \( B_i \) be the arc in the \( x_1, x_2 \) plane, defined in polar coordinates by \( r \leq 1, \ \theta = \pi(1 - 1/i) \). For \( n \) a positive integer, the standard \( n \)-frame \( G_n \) is defined by \( G_n = \bigcup_{i=1}^{n} B_i \). An \( n \)-frame \( F_n \) in \( E_k \) is said to be tame if there is a homeomorphism of \( E_k \) onto itself which carries \( F_n \) onto \( G_n \). Otherwise \( F_n \) is said to be wild. For \( n > 1 \), \( F_n \) is said to be mildly wild if it is wild and \( F_n - (A_i - p) \) is tame for \( i = 1, 2, \ldots, n \).

In [3] it was shown that for each \( n > 1 \) there are mildly wild \( n \)-frames in \( E^2 \). Since there are wild arcs in \( E_k \) for each \( k > 3 \) [1], there will be wild \( n \)-frames for these dimensions. However, we will show that there are no mildly wild \( n \)-frames in \( E_k \) for \( k > 3 \). It then follows that, for \( k > 3 \), the union of two tame arcs meeting only in a common end point is a tame arc. With a small amount of additional argument we will show that a wild arc (simple closed curve) in \( E_k, k > 3 \), must fail to be locally flat at each point of some Cantor set. (If \( S \) is an arc (simple closed curve) in \( E_k \), we say that \( S \) is locally flat at \( p \in S \) if there is a neighborhood \( U \) of \( p \) and a homeomorphism \( h \) which carries \( U \) onto \( E_k \) with \( h(U \cap S) \) lying in the \( x_1 \)-axis.)

Through the remainder of this paper we will assume that we are working in an euclidean space \( E_k \) with \( k > 3 \). We recall that for an arc or simple closed curve in \( E_k \) to be tame it is sufficient that it be locally flat at each of its points. This result for simple closed curves is proved in [5]. The same technique of proof may be used to establish the corresponding result for arcs.

2. Basic lemmas. In [4] it was stated that the result contained in Lemma 2 below followed as a corollary to a theorem concerning manifolds with boundary \( E^{k-1} \) and interior \( E^k \). Since it seems that a more

Presented to the Society, August 29, 1963; received by the editors April 22, 1963.

574
direct proof should be available to the reader, an alternate proof is included in this paper.

**Lemma 1.** Let \( L \) be an arc in \( E^k \), \( p \) an end point of \( L \), and \( U \) a neighborhood of \( L - p \). If \( L \) is locally flat at each point of \( L - p \), then there is a homeomorphism \( f \) of \( E^k \) onto itself such that \( f \) is the identity outside \( U \) and \( f(L) \) is locally polyhedral at each point of \( f(L - p) \).

**Proof.** Let \( p_0 \) be the end point of \( L \), different from \( p \), and let \( p_1, p_2, \ldots \) be a sequence of points of \( L \) converging to \( p \) with \( p_0 < p_1 < p_2 < \cdots \) relative to the order of \( L \) from \( p_0 \) to \( p \). For each integer \( i \) let \( \epsilon_i = 1/i \) and let \( A_i \) be the closed subarc of \( L \) from \( p_0 \) to \( p_i \). Since \( A_2 \) is tame, we may select a closed \( k \)-cell neighborhood \( U \) of \( A_1 \) with the properties: (1) \( U \) is contained in the \( \epsilon_i \)-neighborhood of \( A_1 \) and in \( U \), (2) \( U \cap (L - A_2) = \emptyset \), and (3) \( U \) may be assigned a combinatorial triangulation in which \( A_1 \) is polyhedral. We then apply Homma’s Theorem [5] to obtain a homeomorphism \( f_1 \) of \( E^k \) onto itself such that \( f_1 \) is the identity outside \( U \) and \( f_1(A_1) \) is polyhedral in \( E^k \).

Assume that for each integer \( i > 1 \) certain homeomorphisms \( f_{i-1}, \ldots, f_1 \) of \( E^k \) onto itself have been constructed so that \( f_{i-1} \cdots f_2 \) is polyhedral in \( E^k \). If \( i = 2 \), let \( U_2 \) be a closed \( k \)-cell neighborhood of \( \text{Cl}(f_1(A_2 - A_1)) \) with the properties: (1) \( U_2 \) is contained in the \( \epsilon \)-neighborhood of \( f_1(A_2 - A_1) \) and in \( U \), (2) \( U_2 \cap f_1(L - A_2) = \emptyset \), and (3) \( U_2 \) may be assigned a combinatorial triangulation in which \( U_2 \cap f_1(A_2) \) appears as a polyhedron. We then apply Theorem 2.1 of [5] to obtain a homeomorphism \( f_2 \) of \( E^k \) onto itself such that \( f_2 \) is the identity outside \( U_2 \) and on \( f_1(A_1) \), and \( f_2 \cap f_1 \cap f_2(A_2 - A_1) \) is polyhedral in \( E^k \). Note that at this point \( f_2 f_1(A_2) \) is polyhedral in \( E^k \). If \( i > 2 \), since \( A_{i+1} \) is tame, we may select a closed \( k \)-cell neighborhood \( U_i \) of \( \text{Cl}(f_{i-1} \cdots f_2 f_1(A_1 - A_{i-1})) \) with the properties:

1. \( U_i \) is contained in the \( \epsilon_i \)-neighborhood of \( f_{i-1} \cdots f_2 f_1(A_1 - A_{i-1}) \) and in \( U \),
2. \( U_i \cap [f_{i-1} \cdots f_2 f_1(L - A_{i+1})] = \emptyset \),
3. \( U_i \cap [U_{i-1} - f_i(A_i)] = \emptyset \),
4. \( U_i \cap [U_{i-2} f_{i-1} \cdots f_2 f_1(U_{i+1})] = \emptyset \), and
5. \( U_i \) may be assigned a combinatorial triangulation in which \( U_i \cap f_{i-1} \cdots f_2 f_1(A_2) \) is polyhedral. Again Theorem 2.1 of [5] is applied to obtain a homeomorphism \( f_i \) of \( E^k \) onto itself such that \( f_i \) is the identity outside \( U_i \) and on \( f_{i-1} \cdots f_2 f_1(A_{i-1}) \), and

\[
f_i[\text{Cl}(f_{i-1} \cdots f_2 f_1(A_1 - A_{i-1}))]
\]

is polyhedral in \( E^k \).
For each $x \in E^k$ we set $f(x) = \lim_{i \to +\infty} f_i \cdots f_2 f_1(x)$. Depending principally on the fact that if $x \in \bigcup_{j=1}^n U_j$, $f(x) = x$, and if $x \in U_j$, $f(x) = f_{j+1} f_j f_{j-1}(x)$ one establishes that $f$ is a homeomorphism of $E^k$ onto itself, and $f(L)$ is locally polyhedral at each point different from $f(p)$.

**Lemma 2.** If $L$ is as in Lemma 1, then $L$ is tame.

**Proof.** Let $f$ be a homeomorphism of $E^k$ onto itself so that $f(L)$ is locally polyhedral at each point of $f(L) - f(p)$. We use Lemma 2 of [2] to obtain a homeomorphism $g$ of $E^k$ onto itself such that $gf(L)$ is polyhedral. We then use Theorem 5 of [6] to obtain a homeomorphism $h$ of $E^k$ onto itself that carries $gf(L)$ onto $B_1$.

For each positive integer $n$ we may establish the following theorem.

**Theorem 1.** Let $F_n = \bigcup_{i=1}^n A_i$ be an $n$-frame, with distinguished point $p$, such that $A_i$, $i = 1, 2, \ldots, n$, is locally flat at each point of $A_i - p$. Then $F_n$ is tame.

**Proof.** Theorem 1 has been proved in Lemma 2. We next assume that Theorem 1, $n > 1$, is true and proceed to show that Theorem 1, $n = 1$, is true.

Let $F_n = \bigcup_{i=1}^n A_i$ be an $n$-frame which satisfies the hypotheses of Theorem 1. There is a homeomorphism $\phi_1$ of $E^k$ onto itself such that $F_{n-1} = F_n - (A_n - p)$ is carried onto $G_{n-1}$. Since $\phi_1(A_n)$ is tame, there is a neighborhood $V$ of $\phi_1(A_n - p)$ which does not intersect $G_{n-1}$, and, by Lemma 1, a homeomorphism $\phi_2$ on $E^k$ such that $\phi_2 \phi_1(A_n)$ is locally polyhedral at each point of $\phi_2 \phi_1(A_n - p)$ and $\phi_2$ is fixed outside $V$. We next construct a homeomorphism $\phi_3$ on $E^k$ such that $\phi_3 \phi_2 \phi_1(A_n) = B_n$ and $\phi_3$ is fixed on $G_{n-1}$. The homeomorphism $\phi_3 \phi_2 \phi_1$ will then carry $F_n$ onto $G_n$ and the proof of Theorem 1 will be complete. Since the construction of $\phi_3$ is almost identical with that used in the proof of Lemma 2 of [2], we will only give an outline of the construction.

We use the local connectivity of $\phi_2 \phi_1(A_n)$ to find a sequence $\{V_m\}_{m=1}^\infty$ of closed cubical neighborhoods of the origin such that (1) the end points of $B_n$ and $\phi_2 \phi_1(A_n)$, different from the origin, are outside $V_1$, (2) the diameters of the $V_m$ converge to zero, and (3) if $L$ is any subarc of $\phi_2 \phi_1(A_n)$ whose end points lie in $V_m$, then $L$ is contained in $\text{Int} V_m$.

We will further assume that $\phi_2 \phi_1(A_n) \cap \text{Bd} V_{2m}$ is a finite set of points and that no pair of components of $\phi_2 \phi_1(A_n) - V_{2m}$ share a common end point.

For each positive integer $m$ let $u_{m1}, \ldots, u_{m(k(m))}$ be the components of $\phi_2 \phi_1(A_n) - \text{Int} V_{2m}$ which have both end points on $\text{Bd} V_{2m}$. Since there can be no knotting or linking of polyhedral simple closed curves.

License or copyright restrictions may apply to redistribution; see http://www.ams.org/journal-terms-of-use
in an euclidean space of dimension greater than three, each of the $u_m$ may be moved into $\text{Int } V_{2m}$ without moving points outside $V_{2m-1} - V_{2m+1}$ or on $G_{n-1}$. Thus we may construct a semilinear homeomorphism $f_m$ such that $f_m\phi(A_n) \cap \text{Bd } V_{2m}$ is a single point, and $f_m$ is the identity on $E^k - (V_{2m-1} - V_{2m+1})$ and on $G_{n-1}$. A homeomorphism $f$ is defined by $f(x) = x$, if $x \in E^k - V_1$, $f(x) = f_m(x)$, if $x \in V_{2m-1} - V_{2m+1}$, $m = 1, 2, \cdots$, and $f$ carries the origin onto itself.

Let 0 denote the origin, let $x_0$ be the end point of $B_n$ different from 0, $x_m = B_n \cap \text{Bd } V_{2m}$, $m = 1, 2, \cdots$, $y_0$ the end point of $f\phi(A_n)$ different from 0, and $y_m = f\phi(A_n) \cap \text{Bd } V_{2m}$, $m = 1, 2, \cdots$. Let $g_0$ be a semilinear homeomorphism on $E^k$ such that $g_0$ is fixed on $V_1$ and $g_0(x_0) = x_0$. For $m = 1, 2, \cdots$, let $g_m$ be a semilinear homeomorphism on $E^k$ such that $g_m$ is fixed outside $V_{2m-1} - V_{2m+1}$ and on $G_{n-1}$, and $g_m(y_m) = x_m$. A homeomorphism $g$ is then defined by $g(x) = g_0(x)$, for $x \in E^k - V_1$, $g(x) = g_m(x)$, for $x \in V_{2m-1} - V_{2m+1}$, and $g(0) = 0$.

Again since there can be no knotting or linking of polyhedral simple closed curves in $E^k$, we may construct homeomorphisms $h_m$ with the following properties. The map $h_0$ is fixed on $V_2$ and on $G_{n-1}$, and carries the subarc of $f\phi(A_n)$ from $x_0$ to $x_1$ onto the linear segment $[x_0x_1]$. For $m = 1, 2, \cdots$, $h_m$ is fixed on $E^k - (V_{2m} - V_{2m+2})$ and on $G_{n-1}$, and carries the subarc of $f\phi(A_n)$ from $x_m$ to $x_{m+1}$ onto the linear segment $[x_mx_{m+1}]$. We set $h(x) = h_0(x)$, if $x \in E^k - V_2$, $h(x) = h_m(x)$, if $x \in V_{2m} - V_{2m+2}$, and $h(0) = 0$. Finally we take $\phi_3 = hgf$.

**Corollary 1.** There are no mildly wild $n$-frames in $E^k$.

**Proof.** Suppose $F_n = \bigcup_{i=1}^n A_i$, $n > 1$, is an $n$-frame such that for each $j = 1, 2, \cdots, n$, $F_n - (A_j - p)$ is tame. Then each $A_j$ is tame and, by Theorem 1, $F_n$ is tame.

**Corollary 2.** If $A_1$ and $A_2$ are tame arcs in $E^k$, meeting only in a common end point, then $A_1 \cup A_2$ is tame.

**Theorem 2.** If $A$ is a wild simple closed curve (arc) in $E^k$ and $E$ is the set of points at which $A$ fails to be locally flat, then $E$ contains a Cantor set.

**Proof.** For $A$ a simple closed curve, we know that $E$ is nonempty [5]. By the definition of local flatness, the set of points at which $A$ is locally flat is an open subset of $A$, and $E$ is therefore closed. If we establish that $E$ has no isolated points, there are two possibilities. First, $E$ may be totally disconnected, in which case $E$ is a Cantor set. Secondly, $E$ may have a nondegenerate component $K$, in which case $K$ is either an arc or $K = A$. 

In order to show that there are no isolated points of \( E \), let us consider a point \( q \) of \( A \) such that there is a neighborhood \( U \) of \( q \), relative to \( A \), with \( A \) locally flat at each point of \( U - q \). We select two arcs \( A_1 \) and \( A_2 \) of \( A \) such that \( A_1 \cup A_2 \subset U \) and \( A_1 \cap A_2 = q \). By Lemma 2 and Corollary 2, \( A_1 \cup A_2 \) is tame, and \( A \) is therefore locally flat at \( q \).

This proves Theorem 2 for \( A \) a simple closed curve. A similar argument establishes the theorem for \( A \) an arc.

3. **Added in proof.** Suppose that \( M \) is a finite 1-simplex topologically embedded in \( E^k \), \( k > 3 \), \( V \) is the set of vertices of \( M \), and \( M \) is locally flat at each point of \( M - V \) (equivalently, each 1-simplex of \( K \) is locally flat at each of its interior points). By first applying Homma's Theorem, as in Lemma 1, we may construct a homeomorphism \( f \) of \( E^k \) onto itself such that \( f(M) \) is locally polyhedral at each point of \( f(M) - f(V) \). Then, by applying the technique of proof used in Theorem 1, at each point of \( f(V) \), we may construct a homeomorphism \( g \) of \( E^k \) onto itself such that \( g(f(M)) \) is polyhedral. Thus we see that \( M \) is tame. An immediate corollary is that a finite 1-simplex, topologically embedded in \( E^k \), \( k > 3 \), is tame if and only if each simplex is tame.

**References**

4. P. H. Doyle, *Certain manifolds with boundary which are products*, Mimeographed notes, Virginia Polytechnic Institute, Blacksburg, Va.