A CLASS OF LINEAR TRANSFORMATIONS

R. V. CHACON

1. The purpose of this note is to construct a class of positive and invertible isometries of $L_1(0, 1)$ which give a counterexample in ergodic theory. Specifically, we construct a class such that for each isometry $T$ of the class the limit $\lim_{n \to \infty} (1/n) \sum_{k=0}^{n-1} T^k f$ fails to exist almost everywhere, for some $f \in L_1$. The proof is divided into two lemmas. The first gives that $\liminf_{n \to \infty} (1/n) \sum_{k=0}^{n-1} T^k f = 0$ a.e. The second lemma gives $\limsup_{n \to \infty} (1/n) \sum_{k=0}^{n-1} T^k f = +\infty$ a.e. The second lemma also gives an example of a measurable point transformation having no $\sigma$-finite equivalent invariant measure. This example is considerably simpler than that of [5]. The proof can be modified so that if $\delta > 0$ the transformations $T$ of $L_1(0, 1)$ which are constructed satisfy also the condition that $\|T\|_1 \leq 1 + \delta$.

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2. We give first some definitions and introduce our notation.

Definition 1. Let $A$ and $B$ be measurable sets of the real line of positive and finite measure. The regular map $R$ of $L_1(A)$ onto $L_1(B)$ is the linear extension of the map

$$R \psi_{A_x} = \frac{m(A)}{m(B)} \psi_{B_x},$$

where

$$A_x = (-\infty, x) \cap A, \quad B_x' = (-\infty, x') \cap B,$$

and where $\psi_{A_x}, \psi_{B_x'}$ are the characteristic functions of $A_x$ and $B_x'$, respectively.

It follows that $R$ is a positive and invertible isometry of $L_1(A)$ onto $L_1(B)$ and that there is an invertible point transformation $\tau$ defined almost everywhere mapping $B$ "linearly" onto $A$ such that $R$ is the adjoint of the transformation from $L_\sigma(B)$ onto $L_\sigma(A)$ induced by $\tau$.

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Definition 2. A partition of the unit interval is a finite collection \( \{ J_k, k = 1, \ldots, N \} \) of pairwise disjoint (except for sets of measure zero) sets whose union is the unit interval.

Definition 3. Let \( \{ J_k, k = 1, \ldots, N \} \) be a partition of the unit interval. The regular map \( T \) associated with the partition is the linear operator which is the direct sum of the regular maps of \( L_1(J_k) \) onto \( L_1(J_{k+1}) \), \( k = 1, \ldots, N-2 \), so that \( T \) maps \( L_1(\bigcup_{k=2}^{N-1} J_k) \) onto \( L_1(\bigcup_{k=2}^{N-1} J_k) \).

Note that \( T \) is a positive and invertible isometry.

3. The construction used in the proof has points of contact with the Kakutani “skyscraper” construction as well as with the methods used in [3] and [5]. The proof is based on the following two lemmas. In fact, as remarked in the introduction, Lemma 2 yields an example of the sort given in [5]. To see this, let \( r \) be the point transformation associated with \( T \) and let \( r(x) \) be the Radon-Nikodym derivative of an invariant measure equivalent to Lebesgue measure. The invariance of the measure implies that we may take \( p_k \equiv r \), \( k \geq 0 \) in the theorem of [2] to obtain that \( \lim_{n \to \infty} (1/n) \sum_{k=0}^{n-1} T^k f \) exists a.e. contradicting the fact that the limit superior is infinite.

Lemma 1. Let \( \{ J_k, k = 1, \ldots, N \} \) be a partition of the unit interval and let \( T \) be its associated regular map. If \( f \in L_1(\bigcup_{k=2}^{N-1} J_k) \), \( \epsilon > 0 \) and \( K > 0 \), then there exists a partition \( \{ J_k, k = 1, \ldots, M \} \) such that its associated regular map \( \bar{T} \) is an extension of \( T \), and an \( n > K \) such that

\[
\left| f + \bar{T} f + \cdots + \bar{T}^{n-1} f \right| \leq \epsilon
\]

on a set of measure greater than \( (1 - \epsilon) \).

Proof. We define \( \{ J_k \} \) by setting \( J_k = J_k,J = 1, \ldots, N-1 \), and \( J_k, k = N, \ldots, M \) so that

(i) \( J_k, k = N, \ldots, M \) are pairwise disjoint and \( J_N = \bigcup_{k=1}^{M} J_k \),

(ii) \( m(\bigcup_{k=1}^{M} J_k) \geq (1 - \epsilon/2) \).

It follows that \( \{ J_k, k = 1, \ldots, M \} \) is a partition of the unit interval and that its associated regular map \( \bar{T} \) is an extension of \( T \). Further, for \( M - N \geq N - 2 + j \),

\[
\sum_{k=0}^{N-1+j} \bar{T}^k f = \sum_{k=0}^{N-1} \bar{T}^k f
\]

almost everywhere on \( A = \bigcup_{k=1}^{N-1} J_k \). Since \( \| \bar{T} \| \leq 1 \) it follows that \( \psi_A, \sum_{k=0}^{N-1} \bar{T}^k f \in L_1(A) \), and thus there exists a constant \( H \) with the property that
almost everywhere on a subset $B$ of $A$ of measure $(1 - \epsilon)$. If $M - N \geq N - 2 + \max(K, H/\epsilon)$ then it follows from (1.1) that
\[
\left| \frac{1}{n} \sum_{k=0}^{n-1} T^k f \right| \leq \frac{H}{n} \leq \epsilon
\]
almost everywhere on $B$ for $n = \lceil \max(K, H/\epsilon) \rceil + N$.

Lemma 2. Let $\{J_k, k = 1, \ldots, N\}$ be a partition of the unit interval, and let $T$ be its associated regular map. Let $\epsilon > 0$, $K > 0$, and $B > 0$ be positive constants and $f$ a non-negative function in $L_1(\bigcup_{k=1}^{N-1} J_k)$. Then there exist integers $m_1 \geq K, m_2 \geq K$ and a partition $\{J_k, k = 1, \ldots, M\}$ such that its associated regular map is an extension of $T$ and such that
\[
\sup_{m_1 \leq n \leq m_2} \frac{T^nf}{n} \geq B
\]
on a set of measure greater than $(1 - \epsilon)$.

Proof. We first prove that we may assume without loss of generality
(i) $f = \delta \psi_{J_{k_0}}$ for $\delta > 0$ and $1 \leq k_0 \leq N - 1 - K$,
(ii) $m(\bigcup_{k=1}^{N-1} J_k) \geq 1 - \epsilon/2$,
(iii) $m(J_{N-1}) < \max_{1 \leq k \leq N-2} m(J_k)$.

Since $f \in L_1(\bigcup_{k=1}^{N-1} J_k)$ there exists a set $A_{k_0} \subset J_{k_0}, 1 \leq k_0 \leq N - 1$ such that $f \geq \delta \psi_{A_{k_0}}$, for $\delta > 0$. We then form the partition $\{J_{k}', k = 1, \ldots, 2N\}$ obtained by setting
\[
J'_{k_0} = A_{k_0},
T^{-i}A_{k_0} = J'_{k_0-j}, \quad j = 1, \ldots, k_0 - 1,
T^{i}A_{k_0} = J'_{k_0+j}, \quad j = 1, \ldots, N - 1 - k_0,
J'_N = J_1 - J'_1,
J'_{N+j} = TJ'_N, \quad j = 1, \ldots, N - 2,
\]
and letting $J'_{N-1}$ and $J'_N$ be two disjoint sets such that $J'_{N-1} + J'_N = J_N$ and such that $m(J'_{N-1}) \leq \epsilon/2$. The associated regular map $T'$ is clearly an extension of $T$. Since $T'$ (and $T$) are positive we may take $f = \delta \psi_{A_{k_0}}$.

We suppose in what follows that $f = \delta \psi_{J_{k_0}}$ for some $\delta > 0$ for the given partition $\{J_k, k = 1, \ldots, N\}$, and that $m(\bigcup_{k=1}^{N-1} J_k) \geq 1 - \epsilon/2$. 

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Since $T^{N-k_n-1}f$ is constant on $J_{N-1}$ and zero on its complement, there exists $\delta_1 > 0$ such that $T^{N-k_n-1}f = \delta_1 \psi_{J_{N-1}}$. Let $k_1$ be the least integer such that
\[ m(J_{k_1}) = \max_{1 \leq k \leq N-2} m(J_k), \]
and let $\{A_i, j=1, \ldots, j_0\}$ be pairwise disjoint subsets of $J_{k_1}$ such that, with $\gamma = (1/2B)\delta_1 m(J_{N-1}),$
\begin{align*}
(i) \quad & \sum_{j=1}^{j_0} m(A_j) = \left(1 - \frac{\epsilon}{2}\right) m(J_{k_1}), \\
(ii) \quad & m(A_j) \leq \frac{\gamma}{(j+1)N}, \quad 1 \leq j \leq j_0.
\end{align*}
That $\{A_k\}$ can be chosen to satisfy (2.1) follows easily since
\[ \sum_{j=1}^{j_0} \frac{\gamma}{(j+1)N} = + \infty. \]
We define a partition $\{J_k, k=1, \ldots, (j_0+1)(N-1)+1\}$ as follows: Define
\[ B_0 = c \left( \bigcup_{j=1}^{j_0} T^{-k_1+1} A_j \right) \cap J_{1}, \]
\[ B_j = T^{-k_1+1} A_j, \quad j = 1, \ldots, j_0, \]
and set
\begin{align*}
J_k &= T^{k_1+k} B_0, \quad k = 1, \ldots, N-1, \\
J_k &= T^{k-n} B_{j_0}, \quad k = j_0, \ldots, 2N-2, \\
J_k &= T^{k-(j_0+1)(N-1)+1} B_{j_0}, \quad k = j_0(N-1) + 1, \ldots, (j_0+1)(N-1), \\
J_{(j_0+1)(N-1)+1} &= J_N.
\end{align*}
It follows that the regular map $T'$ of $\{J_k\}$ is an extension of $T$, and that
\begin{equation}
\frac{T^n(\delta_1 \psi_{J_{N-1}})}{N - 1 + n} \geq 2B
\end{equation}
almost everywhere on $J_{k+n-1}, \quad n = 1, \ldots, j_0(N-1).$ Equation (2.2) implies that
\[ \sup_{1 \leq n \leq (j_0+1)(N-1)} (1/n) T^{n-1}f \geq B. \]
almost everywhere on $C_1 = \bigcup_{k=n}^{[k+1]} J_k$. The measure of $C_1$ is, by construction, $(1-\varepsilon/2)$ of the measure of $\bigcup_{k=1}^{[k-1]} J_k$.

In applying Lemma 2 to the invariant measure problem we may assume at the outset that $f > \delta$ and we may therefore obtain a further simplification by omitting the proof of (i), (ii) and (iii).

We state the theorem as follows:

**Theorem.** If $a_1$, $a_2$, are positive constants, then there exists an invertible positive isometry $T$ of $L_1(0, 1)$ and a positive function $f \in L_1 \cap L_\infty$ such that $\|f\|_1 \leq a_1$, $\|f\|_\infty \leq a_2$ and such that

$$\liminf_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} T^k f = 0$$

and

$$\limsup_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} T^k f = \infty$$

almost everywhere.

**Proof.** The theorem follows at once by successive applications of Lemmas 1 and 2 to any initial partition of the unit interval ($\varepsilon \to 0$) and to any function satisfying the norm conditions of the theorem.

**Bibliography**


Brown University