A CLASS OF LINEAR TRANSFORMATIONS

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1. The purpose of this note is to construct a class of positive and invertible isometries of \( L_1(0, 1) \) which give a counterexample in ergodic theory. Specifically, we construct a class such that for each isometry \( T \) of the class the limit \( \lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} T^k f \) fails to exist almost everywhere, for some \( f \in L_1 \). The proof is divided into two lemmas. The first gives that \( \lim \inf_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} T^k f = 0 \) a.e. The second lemma gives \( \lim \sup_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} T^k f = +\infty \) a.e. The second lemma also gives an example of a measurable point transformation having no \( \sigma \)-finite equivalent invariant measure. This example is considerably simpler than that of [5]. The proof can be modified so that if \( \delta > 0 \) the transformations \( T \) of \( L_1(0, 1) \) which are constructed satisfy also the condition that \( \| T \|_{\infty} \leq 1 + \delta \).

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2. We give first some definitions and introduce our notation.

Definition 1. Let \( A \) and \( B \) be measurable sets of the real line of positive and finite measure. The regular map \( R \) of \( L_1(A) \) onto \( L_1(B) \) is the linear extension of the map

\[
R\psi_{A_x} = \frac{m(A)}{m(B)} \psi_{B_x},
\]

where

\[
A_x = (-\infty, x) \cap A, \quad B_{x'} = (-\infty, x') \cap B,
\]

and where \( \psi_{A_x}, \psi_{B_{x'}} \) are the characteristic functions of \( A_x \) and \( B_{x'} \), respectively.

It follows that \( R \) is a positive and invertible isometry of \( L_1(A) \) onto \( L_1(B) \) and that there is an invertible point transformation \( \tau \) defined almost everywhere mapping \( B \) "linearly" onto \( A \) such that \( R \) is the adjoint of the transformation from \( L_\omega(B) \) onto \( L_\omega(A) \) induced by \( \tau \).

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Definition 2. A partition of the unit interval is a finite collection \( \{ J_k, k = 1, \ldots, N \} \) of pairwise disjoint (except for sets of measure zero) sets whose union is the unit interval.

Definition 3. Let \( \{ J_k, k = 1, \ldots, N \} \) be a partition of the unit interval. The regular map \( T \) associated with the partition is the linear operator which is the direct sum of the regular maps of \( L_1(J_k) \) onto \( L_1(J_{k+1}) \), \( k = 1, \ldots, N - 2 \), so that \( T \) maps \( L_1(\bigcup_{k=1}^{N-2} J_k) \) onto \( L_1(\bigcup_{k=2}^{N-1} J_k) \).

Note that \( T \) is a positive and invertible isometry.

3. The construction used in the proof has points of contact with the Kakutani “skyscraper” construction as well as with the methods used in [3] and [5]. The proof is based on the following two lemmas. In fact, as remarked in the introduction, Lemma 2 yields an example of the sort given in [5]. To see this, let \( \tau \) be the point transformation associated with \( T \) and let \( \tau(x) \) be the Radon-Nikodym derivative of an invariant measure equivalent to Lebesgue measure. The invariance of the measure implies that we may take \( p_k = \tau, k \geq 0 \) in the theorem of [2] to obtain that \( \lim_{n \to \infty} (1/n) \sum_{k=0}^{n} T^k f \) exists a.e. contradicting the fact that the limit superior is infinite.

Lemma 1. Let \( \{ J_k, k = 1, \ldots, N \} \) be a partition of the unit interval and let \( T \) be its associated regular map. If \( f \in L_1(\bigcup_{k=2}^{N-1} J_k), \epsilon > 0 \) and \( K > 0 \), then there exists a partition \( \{ J_k, k = 1, \ldots, M \} \) such that its associated regular map \( \tilde{T} \) is an extension of \( T \), and an \( n > K \) such that

\[
\left| \frac{f + \tilde{T}_n f + \cdots + \tilde{T}_n^{-1} f}{n} \right| \leq \epsilon
\]

on a set of measure greater than \((1 - \epsilon)\).

Proof. We define \( \{ J_k \} \) by setting \( J_k = J_k, k = 1, \ldots, N - 1 \), and \( J_k, k = N, \ldots, M \) so that

(i) \( J_k, k = N, \ldots, M \) are pairwise disjoint and \( J_M = \bigcup_{k=1}^{M} J_k \),

(ii) \( m\left( \bigcup_{k=1}^{N} J_k \right) \geq (1 - \epsilon/2) \).

It follows that \( \{ J_k, k = 1, \ldots, M \} \) is a partition of the unit interval and that its associated regular map \( \tilde{T} \) is an extension of \( T \).

Further, for \( M - N \geq N - 2 + j \),

\[
\sum_{k=0}^{N-1+j} \tilde{T}_k f = \sum_{k=0}^{M-1} \tilde{T}_k f
\]

almost everywhere on \( A = \bigcup_{k=1}^{N} J_k \). Since \( \| \tilde{T} \| \leq 1 \) it follows that \( \psi_A, \sum_{k=0}^{N-1} \tilde{T}_k f \in L_1(A) \), and thus there exists a constant \( H \) with the property that
almost everywhere on a subset $B$ of $A$ of measure $(1 - \varepsilon)$. If $M - N \geq N - 2 + \max(K, H/\varepsilon)$ then it follows from (1.1) that

\[
\left| \frac{1}{n} \sum_{k=0}^{n-1} T^k f \right| \leq \frac{H}{n} \leq \varepsilon
\]

almost everywhere on $B$ for $n = [\max(K, H/\varepsilon)] + N$.

**Lemma 2.** Let \( \{J_k, k = 1, \ldots, N\} \) be a partition of the unit interval, and let $T$ be its associated regular map. Let $\varepsilon > 0$, $K > 0$, and $B > 0$ be positive constants and $f$ a non-negative function in $L_1(\bigcup_{k=1}^{N-1} J_k)$. Then there exist integers $m_1 \geq K$, $m_2 \geq K$ and a partition \( \{J_k, k = 1, \ldots, M\} \) such that its associated regular map is an extension of $T$ and such that

\[
\sup_{m_1 \leq n \leq m_2} \frac{T^n - T}{n} \geq \frac{B}{n}
\]

on a set of measure greater than $(1 - \varepsilon)$.

**Proof.** We first prove that we may assume without loss of generality

(i) $f = \delta \psi_{J_{k_0}}$ for $\delta > 0$ and $1 \leq k_0 \leq N - 1 - K$,

(ii) $m(\bigcup_{k=1}^{N-1} J_k) \geq 1 - \varepsilon/2$,

(iii) $m(J_{N-1}) \leq \max_{1 \leq k \leq N - 2} m(J_k)$.

Since $f \in L_1(\bigcup_{k=1}^{N-1} J_k)$ there exists a set $A_{k_0} \subset J_{k_0}$, $1 \leq k_0 \leq N - 1$ such that $f \geq \delta \psi_{A_{k_0}}$ for $\delta > 0$. We then form the partition \( \{J_k', k = 1, \ldots, 2N\} \) obtained by setting

\[
J_{k_0}' = A_{k_0},
\]

\[
T^{-j} A_{k_0} = J_{k_0-j}', \quad j = 1, \ldots, k_0 - 1,
\]

\[
T^j A_{k_0} = J_{k_0+j}', \quad j = 1, \ldots, N - 1 - k_0,
\]

\[
J_N' = J_1 - J_1',
\]

\[
J_{N+j}' = T^j J_N', \quad j = 1, \ldots, N - 2,
\]

and letting $J'_{N-1}$ and $J'_N$ be two disjoint sets such that $J'_{N-1} + J'_N = J_N$ and such that $m(J'_N) \leq \varepsilon/2$. The associated regular map $T'$ is clearly an extension of $T$. Since $T'$ (and $T$) are positive we may take $f = \delta \psi_{A_{k_0}}$.

We suppose in what follows that $f = \delta \psi_{J_{k_0}}$ for some $\delta > 0$ for the given partition \( \{J_k, k = 1, \ldots, N\} \), and that $m(\bigcup_{k=1}^{N-1} J_k) \geq 1 - \varepsilon/2$. 

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Since $T^{N-k_0-1}f$ is constant on $J_{N-1}$ and zero on its complement, there exists $\delta_1 > 0$ such that $T^{N-k_0-1}f = \delta_1 \psi_{J_{N-1}}$. Let $k_1$ be the least integer such that

$$m(J_{k_1}) = \max_{1 \leq k \leq N-2} m(J_k),$$

and let $\{A_j, j=1, \ldots, j_0\}$ be pairwise disjoint subsets of $J_{k_1}$ such that, with $\gamma = (1/2B)\delta_1 m(J_{N-1})$,

\[(i) \sum_{j=1}^{j_0} m(A_j) = \left(1 - \frac{\epsilon}{2}\right) m(J_{k_1}),\]

\[(ii) m(A_j) \leq \frac{\gamma}{(j+1)N}, \quad 1 \leq j \leq j_0.\]

That $\{A_k\}$ can be chosen to satisfy (2.1) follows easily since

$$\sum_{j=1}^{\infty} \frac{\gamma}{(j+1)N} = +\infty.$$

We define a partition $\{J_k, k=1, \ldots, (j_0+1)(N-1)+1\}$ as follows: Define

$$B_0 = c \left( \bigcup_{j=1}^{j_0} T^{-k_1+1} A_j \right) \cap J_1, \quad B_j = T^{-k_1+1} A_j, \quad j = 1, \ldots, j_0,$$

and set

$$J_k^1 = T^{k_1-1} B_0, \quad \quad k = 1, \ldots, N - 1,$$

$$J_k^1 = T^{k_1-N} B_{j_0}, \quad \quad k = N, \ldots, 2N - 2,$$

$$J_k^1 = T^{k_1-(j_0+1)(N-1)+1} B_{j_0}, \quad k = j_0(N-1) + 1, \ldots, (j_0+1)(N-1),$$

$$J_{(j_0+1)(N-1)+1} = J_N.$$

It follows that the regular map $T'$ of $\{J_k^1\}$ is an extension of $T$, and that

$$\frac{T^n(\delta_1 \psi_{J_{N-1}})}{N - 1 + n} \geq 2B$$

almost everywhere on $J_{N+n-1}^1, n=1, \ldots, j_0(N-1)$. Equation (2.2) implies that

$$\sup_{k \leq n \leq (j_0+1)(N-1)} (1/n) T^{n-1} f \geq B$$
almost everywhere on $C^1 = \bigcup_{k=0}^{n+1} J^1_{k}$. The measure of $C^1$ is, by construction, $(1-\epsilon/2)$ of the measure of $\bigcup_{k=1}^{N-1} J_k$.

In applying Lemma 2 to the invariant measure problem we may assume at the outset that $f > \delta$ and we may therefore obtain a further simplification by omitting the proof of (i), (ii) and (iii).

We state the theorem as follows:

**Theorem.** If $a_1, a_2$, are positive constants, then there exists an invertible positive isometry $T$ of $L_1(0, 1)$ and a positive function $f \in L_1 \cap L_\infty$ such that $\|f\|_1 \leq a_1$, $\|f\|_\infty \leq a_2$ and such that

$$\lim \inf_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} T^k f = 0$$

and

$$\lim \sup_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} T^k f = \infty$$

almost everywhere.

**Proof.** The theorem follows at once by successive applications of Lemmas 1 and 2 to any initial partition of the unit interval ($\epsilon \to 0$) and to any function satisfying the norm conditions of the theorem.

**Bibliography**


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