A METHOD FOR CONSTRUCTING DIRICHLET ALGEBRAS

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I. Introduction. We give here some methods for the construction of new Dirichlet algebras out of old ones. The arguments and results are extensions of those we have given in [1]. As one application, we obtain a proper Dirichlet subalgebra of the algebra of functions continuous on the unit circle which extend analytically to the disk, and this answers a question asked in [4].

Related results, arrived at independently, are contained in Glicksberg [3].

We start with a Lemma, perhaps well known, of which a special case was used in [1].

Lemma. Let B be a Banach space, B* its conjugate space. Let U, V be weak-* closed subspaces of B*. Write W for the vector space sum of U and V. Then W is weak-* closed provided there exists a positive constant k such that

\[ \|u\| + \|v\| \leq k\|u + v\|, \quad \text{all } u \text{ in } U, \ v \text{ in } V. \]

Proof. Let \( S = \{ x \in B^*: \|x\| \leq 1 \} \). According to the Krein-Smulian theorem [2, p. 429], it suffices to show \( S \cap W \) is compact (in the weak-* topology). Put \( Q = \{ u + v : u \in U, \ v \in V, \|u\| \leq 1, \|v\| \leq 1 \} \). Evidently \( Q \) is compact. Using our hypothesis, one readily checks that \( S \cap W = S \cap k \cdot Q \), a compact set.

We shall apply this lemma below. First we introduce some notations. Let \( X \) be a compact Hausdorff space, \( C(X) \) the Banach space of all complex-valued continuous functions on \( X \). The conjugate space is identified with the space of complex Baire measures on \( X \). For a closed subalgebra \( A \) of \( C(X) \), we denote by \( A^\perp \) the set of all such measures \( \mu \) satisfying \( \int fd\mu = 0 \), all \( f \in A \). If \( \mu \) is a measure, \( \|\mu\| \) is the total variation of \( \mu \), a positive measure; \( \|\mu\| \) is the norm of \( \mu \) as a linear functional on \( C(X) \). If \( f \in C(X) \), \( f\mu \) is the measure defined by \( (f\mu)(E) = \int_E f d\mu \). If \( \psi \) is a homeomorphism of \( X \) on \( X \), \( \mu \circ \psi \) is the measure defined by \( (\mu \circ \psi)(E) = \mu(\psi(E)) \). If \( A \) is a closed subalgebra of \( C(X) \) which contains the constants and separates the points of \( X \), we shall call it a function algebra on \( X \). If in addition the real parts of functions in \( A \) uniformly approximate to all real continuous functions on \( X \), we call \( A \) a Dirichlet algebra on \( X \). It is easy to see that \( A \) is a

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Dirichlet algebra if and only if $A^\perp$ contains no nonzero real measures. We shall need the following fact (see [5]): If $A$ is any Dirichlet algebra on $X$ and $M$ is any maximal ideal of $A$, there exists a unique positive measure $\sigma_M$ on $X$ with total mass 1, such that

$$\int f d\sigma_M = 0, \quad \text{all } f \in M.$$ 

For a general discussion of Dirichlet algebras, see [5].

If $A$ is any function algebra on a space $X$, we denote by $S(A)$ the space of maximal ideals of $X$, taken in the Gelfand topology and hence a compact Hausdorff space. The space $X$ has a natural homeomorphic embedding in $S(A)$ as a closed subset, and we shall regard $X$ as contained in $S(A)$. $A$ may be regarded as a function algebra on $S(A)$.

If $A$ and $B$ are function algebras on $X$, we shall denote by $S(A) \# S(B)$ the compact space obtained by attaching $S(A)$ to $S(B)$ along $X$, via the natural embeddings of $X$.

**Example.** If $A$ is a function algebra on the circle, and $S(A)$ is the closed disk, $S(A) \# S(A)$ is a 2-sphere (see III below).

**II. General results.**

**Theorem 1.** Let $A$ and $B$ be Dirichlet algebras on $X$. Suppose there is a Baire set $E \subset X$ such that $|\mu|(E) = |\mu|(X - E) = 0$ for every $\mu \in A^\perp$, $\nu \in B^\perp$. Then $A \cap B$ is a Dirichlet algebra on $X$, and $S(A \cap B) = S(A) \# S(B)$.

**Proof.** The hypothesis implies that $\|\mu + \nu\| = \|\mu\| + \|\nu\|$ for every $\mu \in A^\perp$, $\nu \in B^\perp$. Applying the lemma, we find that $A^\perp + B^\perp$ is weak*-closed, and hence $(A \cap B)^\perp = A^\perp + B^\perp$. But if $\mu \in A^\perp$, $\nu \in B^\perp$, and $\mu + \nu$ is real, then $\mu$ and $\nu$ are each real (since $\mu$ and $\nu$ are mutually singular), so $\mu = \nu = 0$. Thus $A \cap B$ is a Dirichlet algebra. To prove the second assertion, we define a map from $S(A) \# S(B)$ into $S(A \cap B)$ as follows: If $M$ is a maximal ideal of $A$ or $B$, $M' = MC \cap BC$ is a maximal ideal of $A \cap B$. It is easy to see that the map: $M \mapsto M'$ is continuous.

To see that the map: $M \mapsto M'$ is injective, recall that to each maximal ideal $M$ there is associated a unique positive measure $\sigma_M$ on $X$, with total mass 1, such that $\int f d\sigma_M = 0$ for all $f \in M$. Evidently, $\sigma_M = \sigma_M'$. If $M = N$, $\sigma_M - \sigma_N$ is a real measure annihilating $A \cap B$, so $\sigma_M = \sigma_N$. If $M$ and $N$ are both ideals of $A$ (or $B$), then it follows that $M = N$. It remains to consider the possibility: $M \in S(A)$, $N \in S(B)$. But if $M \ni S(A) - X$, $\sigma_M(X - E) = 0$. To prove this, we observe that for any $f \in M$, $\int M f^* d\sigma_M = 0$, and so $\int_{X - E} |f| d\sigma_M = 0$.

For each $x \in X$ there is an $f_x \in M$ with $f_x(x) \neq 0$. Using the con-
continuity of the $f_x$ and the compactness of $X$, we get $f_1, \ldots, f_k$ in $M$ such that $F = \sum_{i=1}^k |f_i| > 0$ on $X$. But

$$\int_{X-B} F d\sigma_M = \sum_{i=1}^k \int_{X-B} |f_i| d\sigma_M = 0$$

which implies that $\sigma_M(X-E) = 0$. Similarly, if $N \subseteq S(B)-X$, $\sigma_N(E) = 0$. Since $\sigma_M = \sigma_N$, $\sigma_M(X) = 0$, which is false. It follows that $M$ and $N$ correspond to the same point of $X$. Thus the map $M \rightarrow M'$ is injective.

To see that the map is surjective, let $L$ now denote any maximal ideal of $A \cap B$. We must show that $L$ is contained in a maximal ideal of either $A$ or $B$. Put $\sigma_L = \phi_1 + \phi_2$ where $\phi_1(Z) = \sigma_L(Z \cap E)$ for any Baire set $Z \subseteq X$. Then $\phi_1$ and $\phi_2$ are positive measures, not both zero. Suppose $\phi_1 \neq 0$. We then assert that $L$ is contained in a maximal ideal of $A$. To prove this, it suffices to show that the ideal of $A$ generated by $L$ is proper. Now for any $f \in L$, $f\sigma_L \in (A \cap B)^+$, so $f\phi_1 + f\phi_2 = \mu + \nu$, where $\mu \in A^+$ and $\nu \in B^+$. Since $|f\phi_1 - \mu|(X-E) = |\nu - f\phi_2|(E)$, we conclude that $f\phi_1 = \mu$. Thus $\int g d\phi_1$ for every $f \in L$, $g \in A$. Since $\int d\phi_1 > 0$, it follows that the ideal generated by $L$ in $A$ is proper.

Let $A$ be a Dirichlet algebra on $X$, and let $\psi$ be a homeomorphism of $X$ on itself. We define $A(\psi) = \{f \in A : f \circ \psi \in A\}$. $A(\psi)$ is clearly a closed subalgebra of $A$ containing the constants. It may, of course, reduce to the constants.

We call $\psi$ singular (with respect to $A$) if there exists a Baire set $E \subseteq X$ such that

$$(*) \quad |\mu|(X-E) = |\mu|(\psi^{-1}(E)) = 0, \quad \text{all} \ \mu \in A^+. $$

**Theorem 2.** If $\psi$ is singular, $A(\psi)$ is a Dirichlet algebra on $X$. Moreover, $S(A(\psi)) = S(A) \# \psi S(A)$, the space obtained by attaching $S(A)$ to $S(A)$ along $X$ via the map $\psi$; and $A(\psi)$ is a proper subalgebra of $A$, unless $A = C(X)$.

**Proof.** Let $B = \{f \in C(X) : f \circ \psi \in A\}$. Since $A$ is a Dirichlet algebra on $X$, so is $B$, and clearly $A(\psi) = A \cap B$. Now $\nu \in B^+$ if and only if $\int f \circ \psi^{-1} d\nu = 0$ for all $f \in A$, and so if and only if $\nu \circ \psi \in A^+$. Let $E$ be the set satisfying $(*).$ Then $|\mu|(X-E) = 0$ for all $\mu \in A^+$; also if $\nu \in B^+$, $|\nu|(E) = |\nu \circ \psi|(\psi^{-1}(E)) = 0$, since $\nu \circ \psi \notin A^+$. Thus Theorem 1 applies to the algebras $A$ and $B$, yielding that $A(\psi) = A \cap B$ is a Dirichlet algebra on $X$, and that $S(A(\psi)) = S(A) \# S(B) = S(A) \# \psi S(A)$. If $A(\psi) = A$, then $A \subseteq B$, so $A^+ \supseteq B^+$ so for every $\nu \in B^+$, $|\nu|(X-E) = |\nu|(E) = 0$, thus $\nu = 0$, and hence $B = C(X)$, whence $A = C(X)$.

Suppose next that $A$ is a function algebra on $X$ and $G$ a group of homeomorphisms of $X$ on itself, such that $f \circ g \in A$, for every $f \in A$. 

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In a natural way each \( g \in G \) extends to a homeomorphism of \( S(A) \) on itself: if \( M \) is a maximal ideal of \( A \), \( g(M) = \{ f \in A : f \circ g \in M \} \).

Put

\[ A' = \{ f \in A : f = f \circ g \text{ for every } g \in G \}. \]

Then \( A' \) is a uniformly closed algebra of functions on \( X \). We denote by \( X/G \) the identification space induced by \( G \). \( A' \) may be regarded as an algebra of functions on \( X/G \).

**Theorem 3.** Suppose \( G \) is finite. Then \( S(A') = S(A)/G \). If \( A \) is a Dirichlet algebra on \( X \), then \( A' \) is a Dirichlet algebra on \( X/G \).

**Proof.** We may regard \( A' \) as an algebra of functions on \( S(A)/G \). Thus there is an obvious continuous map \( \tau \) from \( S(A)/G \) into \( S(A') \).

To see that \( \tau \) is injective, let \( m_1, m_2 \) be points of \( S(A) \) giving rise to distinct points of \( S(A)/G \), i.e., such that \( g(m_1) \neq g(m_2) \) for all \( g \in G \). Choose \( f \in A \) such that \( f(g(m_1)) \neq 0 \) for all \( g \in G \), \( f(m_2) = 0 \). Put \( F = \prod_{g \in G} f \circ g \). Then \( F \in A', F \circ g = F \) for every \( g \in G \), so \( F \in A' \), and \( F(m_1) \neq 0 = F(m_2) \). Thus \( \tau(m_1), \tau(m_2) \) are distinct elements of \( S(A') \).

To show that \( \tau \) is surjective, consider a maximal ideal \( M \) of \( A' \) and let \( M_1 \) denote the ideal generated by \( M \) in \( A \). Suppose \( M_1 \) is not proper. Then we can find \( f_1, \ldots, f_n \) in \( M \) and \( k_1, \ldots, k_n \) in \( A \) with

\[ \sum_{i=1}^{n} f_i k_i = 1. \]

Put \( k'_i = \sum_{g \in G} k_i \circ g \). Then \( k'_i \in A' \) and

\[ \sum_{i=1}^{n} f_i k'_i = N = \text{order of } G. \]

Hence \( M \) is the unit ideal in \( A' \), a contradiction. Thus \( M_1 \) is proper, and \( \tau \) maps the point of \( S(A)/G \) induced by \( M_1 \) on \( M \). Thus \( \tau \) is surjective, and so \( S(A)/G = S(A') \).

We now let \( A \) be a Dirichlet algebra on \( X \). Let \( u \) be a real continuous function on \( X \) such that \( u \circ g = u \) for all \( g \in G \), so that \( u \) can be regarded as defined on \( X/G \). Let \( \epsilon > 0 \) be arbitrary. Since \( A \) is a Dirichlet algebra, there exists \( f \) in \( A \) such that \( \| \Re f - u \| < \epsilon \), (where \( \| \| \) denotes the maximum modulus on \( X \)).

Put \( F = (1/N) \sum_{g \in G} f \circ g \). Clearly \( F \in A' \) and \( \| \Re F - u \| < \epsilon \), since \( \| \Re f \circ g - u \| < \epsilon \) for each \( g \in G \). Hence \( A' \) is a Dirichlet algebra on \( X/G \).

**Note.** In the latter part of the theorem, the same argument can be made if \( G \) is compact (instead of finite), with integration over \( G \) re-
placing summation. Also, the same method shows that if $A$ is a maximal subalgebra of $C(X)$, then $A'$ is a maximal subalgebra of $C(X/G)$.

III. Applications. Let $\Gamma$ denote the unit circle $|z| = 1$ in the $z$-plane, and $A_0$ the algebra of all continuous functions on $\Gamma$ which admit continuous extensions to $|z| \leq 1$, analytic on $|z| < 1$. It is well known that: $A_0$ is a Dirichlet algebra on $\Gamma$; $S(A_0)$ can be identified with the closed disk $|z| \leq 1$; every measure in $A_0^\ast$ is absolutely continuous with respect to Lebesgue measure on $\Gamma$ (F. and M. Riesz); $A_0$ is a maximal subalgebra of $C(\Gamma)$ (see [5]).

Let $\psi$ be a homeomorphism of $\Gamma$ on $\Gamma$ such that for some Borel set $E$ of Lebesgue measure $2\pi$, $\psi^{-1}(E)$ has Lebesgue measure zero. We shall call $\psi$ singular. Recall that by definition, $A_0(\psi) = \{f \in A_0 | f \circ \psi \in A_0\}$. In view of the facts summarized above, Theorem 2 applies, to give:

**Corollary 1.** $A_0(\psi)$ is a Dirichlet algebra on $\Gamma$, and is a proper subalgebra of $A_0$. $S(A_0(\psi))$ is a 2-sphere.

Let $\psi$ be a singular homeomorphism of $\Gamma$ on itself such that $\psi \circ \psi = \text{identity}$. Clearly $f \circ \psi \in A_0(\psi)$ for every $f \in A_0(\psi)$. Put

$$A_\psi = \{f \in A_0 | f = f \circ \psi\}.$$ 

Let $G$ be the two element group generated by $\psi$. Then $A_\psi = \{f \in A_0(\psi) | f = f \circ g \text{ for every } g \in G\}$. Thus Theorem 3 can be applied, with $A = A_0(\psi)$ and $A' = A_\psi$. We get, first

**Corollary 2.** $A_\psi$ is a Dirichlet algebra on $\Gamma/\psi$.

The topology of $S(A_\psi)$ and of $\Gamma/\psi$ depends on whether or not $\psi$ preserves orientation on $\Gamma$.

**Corollary 3. If $\psi$ reverses orientation, $\Gamma/\psi$ is (homeomorphic to) a closed interval and $S(A_\psi)$ is a 2-sphere. If $\psi$ preserves orientation and has no fixed points, $\Gamma/\psi$ is a circle and $S(A_\psi)$ is a (real) projective plane.**

**Proof.** The assertions for $\Gamma/\psi$ are easily verified. By Theorem 3, $S(A_\psi) = S(A_0(\psi))/G$. Since $S(A_0(\psi))$ is a 2-sphere, $S(A_0(\psi))/G$ is easily seen to be a closed disk with the boundary identification induced by $\psi$. From this the assertions follow.

**Note.** The situation when $\psi$ reverses orientation was described in [1]. The method of proof of the following result was used in the corresponding theorem in [1].

**Theorem 4.** $A_\psi$ is a maximal subalgebra of $C(\Gamma/\psi)$. 
Proof. Let $B$ be a closed subalgebra of $C(\Gamma/\psi)$. We may regard $B$ as a closed subalgebra of $C(\Gamma)$ such that $f = f \circ \psi$ for every $f \in B$. Assume $A_\psi \subset B$. Let $\mathcal{O}$ denote the space of all measures $\nu$ on $\Gamma$ such that $\nu = -\nu \circ \psi$, so $\mathcal{O} = C(\Gamma/\psi)^\perp$ under the obvious identifications. If $\mu \in A_\psi^*$ and $\nu \in \mathcal{O}$, we have

$$
\|\mu\| = \frac{1}{2} \|\mu + \mu \circ \psi\| = \frac{1}{2} \|\mu + \nu + \mu \circ \psi + \nu \circ \psi\| \\
\leq \frac{1}{2} (\|\mu + \nu\| + \|\mu \circ \psi + \nu \circ \psi\|) = \|\mu + \nu\|.
$$

Hence by the Lemma $A_\psi^* + \mathcal{O}$ is weak-* closed, and therefore $= A_\psi^\perp$. Thus if $\lambda \in B^\perp$, $\lambda = \mu + \nu$ for some $\mu \in A_\psi^\perp$, $\nu \in \mathcal{O}$. If $B \neq C(\Gamma/\psi)$, then we have such a $\lambda$ outside $\mathcal{O}$, so $0 \neq \mu = \lambda - \nu \in A_\psi^* \cap B^\perp$. Now for any $f \in B$, $f_\mu \in B^\perp$, so $f_\mu = \mu_1 + \nu_1$ for some $\mu_1 \in A_\psi^\perp$, $\nu_1 \in \mathcal{O}$. Thus $\mu_1 - f_\mu = f_\mu \circ \psi - \mu_1 \circ \psi$. Since $\psi$ is singular, $f_\mu - \mu_1 = 0$, or $f_\mu \in A_\psi^\perp$. Thus $\int g d\mu = 0$ for every $g \in A_\psi$, $f \in B$. Since $\mu \neq 0$ and $A_\psi$ is a maximal algebra, this implies $B \subset A_\psi$, and thus $B = A_\psi$. Hence $A_\psi$ is maximal.

Appendix. As an application of the algebras $A_\psi(\Psi)$ introduced above, we now give a closure result on the unit circle $\Gamma$. Let $\Psi$ be a homeomorphism of $\Gamma$ which reverses orientation on $\Gamma$. We do not assume here that $\Psi$ is singular.

Theorem 5. Every continuous function on $\Gamma$ can be uniformly approximated by linear combination of powers $z^n$, $n \geq 0$, and $\Psi^n$, $n \geq 0$.

The proof makes use of arguments given in [1]. Let $f \in A_\psi(\Psi)$. Let $\alpha$ be a value taken by $f$ in $|z| < 1$. Suppose $\alpha \neq f(\Gamma)$. Put $g = f - \alpha$. Then $g \in A_\psi(\Psi)$ and $\var\arg g > 0$. But $\var\arg g = -\var\arg g(\Psi)$, since $\Psi$ reverses direction, and $\var\arg g(\Psi) \geq 0$, since $g(\Psi) \in A_\psi$. This is a contradiction, and so $\alpha \in f(\Gamma)$. Thus $f( |z| < 1) \subset f(\Gamma)$. We conclude that, unless $f$ is a constant, $f(\Gamma)$ has positive area in the plane.

Let $\mu$ be any measure on $\Gamma$ with $\mu \perp z^n$, $n \geq 0$, and $\mu \perp \Psi^n$, $n \geq 0$. All we need to do is to show that $\mu$ must be 0. By the F. and M. Riesz theorem, $d\mu = h(t)dt$ on $(-\pi, \pi)$, where there is some $\eta$ in the Hardy class $H^1$ with $\eta(0) = 0$ and $\eta(e^{it}) = h(t)$. Put $H(\theta) = \int_{-\pi}^\pi h(t)dt$. It is easy to verify that $H \in A_\psi$.

Because we may rotate it, that $f(\Gamma)$ has no loss of generality to assume that $\Psi(-1) = -1$. We can then set: $\psi(e^{it}) = e^{i\eta(t)}$, where $\psi$ is a strictly decreasing continuous function on $(-\pi, \pi)$ with $\psi(-\pi) = \pi, \psi(\pi) = -\pi$. Now

$$
\int_{-\pi}^\pi h(t)e^{i\eta(t)}dt = 0, \quad n \geq 0.
$$
Integrating by parts, we get
\[ \int_{-\pi}^{\pi} H(t) d(e^{i\psi(t)}) = 0, \quad n \geq 0. \]
Putting \( u = \psi(t) \) in this integral, we have
\[ \int_{-\pi}^{\pi} H(\psi^{-1}(u)) d(e^{iu}) = 0, \quad n \geq 0. \]
Hence \( H(\psi^{-1}) \subset A_\Phi \). Thus \( H \subset A_\Phi(\Psi^{-1}) \). Since \( \Psi^{-1} \) also reverses orientation, we conclude by the above that either \( H \) is constant or \( H(\Gamma) \) has positive area. But \( H \) is absolutely continuous. Hence \( H \) is constant and so 0, whence \( \mu = 0 \), and we are done.

References


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