ONE-SIDED INVARIANT SUBSPACES AND DOMAINS OF UNIQUENESS FOR HYPERBOLIC EQUATIONS

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1. Introduction. Suppose \( t \to U(t) \) is a continuous one-parameter group of unitary operators on a complex Hilbert space \( K \), and let \( H \) be the self-adjoint generator of this group. Do there exist real or complex linear closed subspaces of \( K \) which are invariant under the semigroup \( [U(t) | t \geq 0] \) but which are not invariant under the full group? In \( \S 2 \) we investigate this question under the additional hypothesis that \( H \) be a positive-definite operator. Our basic result, when \( H \geq cI > 0 \), is that there are no proper one-sided invariant manifolds; the invariant subspaces (real or complex linear) for \( [U(t) | -\infty < t < \infty ] \) are precisely those for \( [U(t) | t < t_0 < \infty ] \), \( t_0 \) being an arbitrary real number. This fact is exploited in \( \S 3 \) to obtain some sharp results on domains of uniqueness for normalizable (finite-energy) solutions of the Klein-Gordon and related hyperbolic partial differential equations. The general result is that such solutions are uniquely determined by their values on an open time-like backward cone in space-time. This result carries over to the quantized Klein-Gordon field (see Segal [5]), and it follows that the collection of field operators \( R(f) \), with \( f \) a testing function supported on an open time-like backward cone, is complete, i.e. bounded functions of these operators are weakly dense in the space of all bounded operators on the field state space.

Lax, Morawetz and Phillips [3] have recently considered scattering for the wave equation, which is the limiting case of the Klein-Gordon equation when the mass \( m \to 0 \). An interesting result (cf. our Theorem 3.1) of their investigation is that a finite-energy solution of the wave equation which vanishes in both the forward and backward light cones is zero identically.

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2. Groups with positive generators. Suppose \( t \rightarrow U(t) \) is a one-parameter unitary group on a complex Hilbert space \( K \). By Stone's theorem \([4]\) there exists a self-adjoint operator \( H \) on \( K \) with spectral family \([E_x]\) such that \( U(t) = \exp iHt \).

**Theorem 2.1.** If \( H \) is a strictly positive operator, then any closed real-linear manifold \( M \) in \( K \) which is invariant under \( \{ U(t) \mid t \leq t_0 \} \) is also invariant under \( \{ U(t) \mid t < t_0 \} \).

**Proof.** As a preliminary remark, we observe that if \( (u, v) \) is the complex inner product on \( K \), then \( K \) becomes a real Hilbert space \( K_r \) under the inner product \( [u, v] = \Re(u, v) = \frac{1}{2}[(u, v) + (v, u)] \). We shall use the symbol \( \perp \) to denote orthogonality in \( K_r \).

If \( u, v \) are in \( K \), then by virtue of the positivity of \( H \),

\[
(U(t)u, v) = \int_{-\infty}^{\infty} e^{ims}d(E_m u, v),
\]

with \( m \) a fixed positive number. This representation makes it evident that \( f(t) = (U(t)u, v) \) can be extended to a holomorphic function in the half-plane \( |t + is| > 0 \). Furthermore, \( |f(t + is)| \leq e^{-ms}||u|| \cdot ||v|| \).

Suppose now that \( u \) belongs to \( M \), and \( v \) belongs to \( M^\perp \). By hypothesis \( U(t)u \) also belongs to \( M \), if \( t \geq t_0 \), hence \( \Re f(t) = 0, t \geq t_0 \). To complete the proof of Theorem 2.1 it suffices to show that such an \( f \) must be identically zero. For, if \( f \equiv 0 \), then \( v \perp u \) implies \( v \perp U(t)u \) for all \( t \). \( M \) is a closed subspace of \( K_r \), so that for any \( t \),

\[
U(t)M \subset (M^\perp)^\perp = M.
\]

Q.E.D.

It remains to prove the

**Lemma.** Let \( f \) be analytic for \( \Im z > 0 \), continuous for \( \Im z \geq 0 \). Suppose further that \( |f(x + iy)| \leq Ce^{-mx} \) for some \( m > 0 \) and all \( x \). Then \( \Re f(x) = 0 \) for all \( x > t_0 \) implies that \( f = 0 \).

**Proof.** Under the hypotheses we may apply the Schwartz reflection principle to continue \( f \) analytically as a bounded function in \( \Re z > t_0 \) by defining \( f(x - iy) = -f(x + iy)^* \). It follows that \( f(t_0 + iy) = O(e^{-mt_0}) \). By a Phragmen-Lindelöf theorem \([8, \S5.8]\), \( f(z) = 0 \) identically in the half-plane, \( \Re z \geq t_0 \), hence also in the upper half-plane.

3. Klein-Gordon equation and domains of uniqueness. An important physical system in relativistic quantum mechanics is the quantized scalar meson field \([6]\), whose state space of inputs and outputs (the asymptotic free field) is built up from normalizable solutions of the Klein-Gordon equation.
\[ \Box \phi = m^2 \phi, \]
where \( m > 0 \) and \( \Box \) is the wave operator \( \Delta - \partial^2/\partial t^2 \). The Hilbert space \( K_m \) of real normalizable solutions of this equation may be described most succinctly via Fourier analysis as follows (see [5]): Identifying \( x_0 \) with \( t \), we let \( x = (x_0, x_1, \cdots, x_n) \) denote a vector in \( \mathbb{R}^{n+1} \) and \( k = (k_0, k_1, \cdots, k_n) \) a vector in the dual space, with \( k \cdot x = k_0 x_0 - k_1 x_1 - \cdots - k_n x_n \). Consider the complex-valued functions on the hyperboloid \( k \cdot k = m^2 \) which are measurable and square-summable with respect to the Lorentz-invariant measure \( d\chi(k) = |k_0|^{-1} d^n k, \) \( d^n k \) denoting \( n \)-dimensional Lebesgue measure \( dk_1 dk_2 \cdots dk_n \). To such a function \( f \) corresponds a (generalized) solution \( \phi \) given by

\[ (3.1) \quad \phi(x) = \int e^{i k \cdot x} f(k) d\chi(k). \]
So that \( \phi \) be real-valued, we require that \( f(-k) = f(k)^* \); \( K_m \) is then the real Hilbert space of all such Hermitian-symmetric, square-summable \( f \), with inner product

\[ [f, g] = \int f(k) g(-k) d\chi(k), \]
which is always real.

The invariance of the wave operator \( \Box \) under the inhomogeneous Lorentz group yields an orthogonal representation of this group on \( K_m \); in particular, time translations, \( x \rightarrow x + t e_0, \) \( e_0 \) a unit vector on the \( x_0 \)-axis, give rise to a one-parameter orthogonal group

\[ U(t) : f(k) \rightarrow e^{i t k_0} f(k), \quad f \in K_m. \]

A distinguished property of this representation, for \( m > 0 \), is that a complex structure may be put on \( K_m \) such that the representation of the Lorentz group becomes unitary, and \( U(t) = e^{i t H} \) with self-adjoint generator \( H \geq m \). (See Segal [6].)

Explicitly, let \( j \) be the orthogonal transformation on \( K_m \) sending \( f(k) \rightarrow \text{sgn}(k_0) f(k) \); then \( j^2 = -I \), and \( j \) commutes with \( U(t) \). By defining a complex inner product

\[ (f, g) = [f, g] - i [jf, g] \]
and multiplication by complex scalars

\[ (\alpha + i \beta) f = \alpha f + \beta j f, \]
we make \( K_m \) into a complex Hilbert space. The operator \( H \) is given
by multiplication by \(|k_0|\), and since \(k \cdot k = m^2\), it follows that the spectrum of \(H = \text{range } |k_0| = [m, \infty)\).

**Theorem 3.1.** If \(\phi\) is a real-valued normalizable solution of \(\Box \phi = m^2\phi\), vanishing on an open time-like cone, then \(\phi = 0\) identically.

**Proof.** By a time-like cone is meant a cone in \(\mathbb{R}^{n+1}\) containing a time-like vector \(x (x_0^2 > x_1^2 + \cdots + x_n^2)\). Since \(K_m\) is invariant under a change of coordinates in \(\mathbb{R}^{n+1}\) given by a Lorentz transformation, it is enough to consider the case \(\phi = 0\) on an open cone \(C\) containing the negative \(x_0\)-axis. If \(\phi\) corresponds to \(\psi\) via (3.1), then the hypothesis is equivalent to \([\psi, P\Psi] = 0\) for every real-valued function \(\Psi \in C_0^\infty (\mathbb{R}^{n+1})\) with compact support in \(C\). \(P\Psi\) here denotes the projection of \(\Psi\) onto \(K_m\), i.e. the restriction of the Fourier transform of \(\Psi\) (in \(\mathbb{R}^{n+1}\)) to the hyperboloid \(k^2 = m^2\).

Let now \(M = [P\Psi | \Psi \in C^\infty_0 (\mathbb{R}^{n+1}), \text{Supp} (\Psi) \subset C]\), the bar signifying closure in \(K_m\). By continuity \([f, g] = 0\) for all \(g \in M\); furthermore, \(M\) is invariant under \([U(t)| \ t \geq 0]\), since \(U(t)P\Psi = P\Psi_t\), with \(\Psi_t(x) = \Psi(x + te_0)\). Thus \(M\) is a one-sided invariant manifold and by Theorem 2.1 we conclude that \(U(t)M \subset M\) for all \(t\). For any \(\Psi \in C^\infty_0 (\mathbb{R}^{n+1})\), however, there exists a \(t > 0\) such that \(\text{Supp} (\Psi_t) \subset C\). Hence \(P\Psi = U(-t)P\Psi_t \in M\), and so \([f, P\Psi] = 0\) for all test functions \(\Psi\), implying that \(f = 0\).

It follows from the proof that we have

**Corollary 3.1.** If \(M\) is the set of all \(C^\infty\) functions on space-time with compact support contained in a fixed open time-like cone, then the projection of \(M\) onto \(K_m\) is dense in \(K_m\).

**Remarks.** Theorem 3.1 is a sharp result in two directions. In the \(m = 0\) case, where the energy operator is not strictly positive, there exist nonzero normalizable solutions to the wave equation,

\[\Box \phi = 0,\]

which vanish in the backward light cone. Furthermore, nonzero \(C^\infty\) solutions of the Klein-Gordon equation exist which vanish in the backward cone by virtue of familiar general principles concerning hyperbolic equations. (See Courant-Hilbert [1, pp. 450–459] for a discussion of this characteristic Cauchy problem.) Thus we see that the physically-motivated requirements of normalizability and positivity of the energy force a solution to be determined everywhere by its values on any open time-like cone.

**4. Klein-Gordon equation with perturbations.** Our preceding result on domains of uniqueness for normalizable solutions of the KG
(Klein-Gordon) equation is also valid for a class of linear time-independent perturbations, and nonlinear time-dependent perturbations of the equation. In the first case, where the perturbation consists of a non-negative potential $V(x)$, an abstraction of the proof used for the KG equation establishes the result. In the second case, where the perturbation is, e.g., a continuous time-dependent operator small at $t = \pm \infty$, results of Walter Strauss [7] on nonlinear scattering allow the result for the KG equation to be used, but with the weaker conclusion that only the full backward light cone is a domain of uniqueness for normalizable solutions of the nonlinear equation.

Let us consider first the KG equation with potential, viz.,

\[(4.1) \Box \phi = (m^2 + V)\phi,\]

where $V$ is an a.e. non-negative measurable function of the space variable $x$, and $m^2 > 0$. To avoid irrelevant complications, we shall assume that any singularities of $V$ are mild enough so that the operator $A_0 = -\Delta + m^2 + V$, with $D(A_0) = S$ (the Schwartz space of rapidly decreasing functions) is essentially self-adjoint on $K$, the real Hilbert space of real-valued Lebesgue square-summable functions on $\mathbb{R}^n$ (see Kato [2]). We denote the closure of $A_0$ by $A$; $A$ is then self-adjoint and is easily seen to satisfy $A \geq m^2 I$. By the spectral theorem $B = A^{1/2}$ exists as a positive self-adjoint operator.

To obtain a Hilbert space structure on solutions of (4.1), we introduce the real Hilbert space $\mathcal{X}_0 = D(A) \oplus D(B)$, with the inner product

\[(u, v)_{\mathcal{X}_0} = (Au_1, Av_1)_K + (Bu_2, Bv_2)_K\]

when

\[u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \quad \text{and} \quad v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}.\]

Writing (4.1) in the abstract form

\[(4.2) \frac{d^2 \phi}{dt^2} = - A \phi,\]

we define the space $\mathcal{X}$ of normalizable solutions of (4.2) as the set of all $K$-valued functions of $t$ satisfying (i) $t \rightarrow \phi(t)$ is strongly differentiable and the derivative $\phi'(t)$ is absolutely continuous and a.e. strongly differentiable, (ii) $\phi(0) \in D(A)$ and $\phi'(0) \in D(B)$, (iii) $\phi$ satisfies (4.2) a.e.

**Lemma 4.1.** Every element of $\mathcal{X}$ has the unique representation
(4.3) \[
\begin{pmatrix}
\phi(t) \\
\phi'(t)
\end{pmatrix}
= 
\begin{pmatrix}
\cos tB & B^{-1} \sin tB \\
-B \sin tB & \cos tB
\end{pmatrix}
\begin{pmatrix}
\phi(0) \\
\phi'(0)
\end{pmatrix}.
\]

(See Strauss [7].)

From Lemma 4.1 we see that each element \( \phi \) of \( \mathcal{C} \) with Cauchy data \( \phi(0) = \phi_0, \phi'(0) = \phi_1 \) at time \( t = 0 \) corresponds uniquely to the element

\[
\begin{pmatrix}
\phi_0 \\
\phi_1
\end{pmatrix}
\]

of \( \mathcal{C}_0 \). If the matrix operator in (4.3) is denoted by \( U_t \), then \( t \rightarrow U_t \) is a continuous one-parameter orthogonal group on \( \mathcal{C}_0 \) that maps the Cauchy data of \( \phi \) at \( t = 0 \) into the Cauchy data of \( \phi \) at time \( t \).

**Theorem 4.1.** If \( \phi \) is a normalizable solution of (4.2) and \( \mathcal{C} \) is an open cone in \( \mathbb{R}^{n+1} \) with vertex on the \( t \)-axis and containing a semi-infinite segment of the \( t \)-axis, then \( \phi = 0 \) on \( \mathcal{C} \) implies \( \phi = 0 \).

**Proof.** By hypothesis

(4.4) \[
\langle \phi, u \rangle = \int_{\mathbb{R}^{n+1}} \phi(x, t) u(x, t) dx dt = 0
\]

for all \( u \in C^\infty_c(\mathcal{C}) \), and the conclusion of the theorem follows if we show that (4.4) must then hold for all \( u \in C^\infty_c(\mathbb{R}^{n+1}) \). For this purpose we need the following construction:

Let \( J \) be the operator on \( \mathcal{C} \) which acts formally as the Hilbert transform with respect to time, i.e.

\[
J(\cos tB\phi_0 + B^{-1} \sin tB\phi_1) = -\sin tB\phi_0 + B^{-1} \cos tB\phi_1.
\]

Equivalently, in terms of its action on the space \( \mathcal{C}_0 \) of Cauchy data at \( t = 0 \), \( J \) corresponds to the matrix operator

\[
J = \begin{pmatrix} 0 & B^{-1} \\ -B & 0 \end{pmatrix}.
\]

**Lemma 4.2.** (a) \( J \) is an isometric transformation on \( \mathcal{C}_0 \) satisfying \( J^* = -J \), \( J^2 = -I \).

(b) \( JU_t = U_t J \).

**Proof.** Direct calculation.

By Lemma 4.2, just as in the case of the KG equation in §3, we can define a complex structure on \( \mathcal{C}_0 \) via \( J \), and \( t \rightarrow U_t \) becomes a one-parameter unitary group on the complex Hilbert space \( \mathcal{C}_0 \). The self-adjoint generator \( H \) of \( U_t \) is obtained as
\[
\lim_{t \to 0} (jt)^{-1}(U_t - I) = \begin{pmatrix} B & 0 \\ 0 & B \end{pmatrix},
\]
with \( D(H) = D(B^1) \oplus D(B^2) \). Since \( H \) commutes with \( j \), and \( B \geq mI \) on \( K \), it follows that \( H \geq mI \) on \( \mathcal{K}_e \).

We now return to the proof of Theorem 4.1. From Theorem 2.1 and the construction above, it follows that the real-linear span in \( \mathcal{K}_0 \) of the Cauchy data of \( \phi \) at times \( t \) belonging to any semi-infinite interval coincides with the span of the Cauchy data of \( \phi \) at all times \(- \infty < t < + \infty\). Since \( \phi \to \langle \phi, u \rangle \) (defined in Equation (4.4)) is a continuous linear functional on \( \mathcal{K}_0 \), we conclude, as in the proof of Theorem 3.1, that \( \langle \phi, u \rangle = 0 \) for all \( u \in C^*_0(\mathcal{C}_e), - \infty < t < \infty \), where \( \mathcal{C}_e \) is the translated cone \( te_0 + \mathcal{C} \). Any compact set in \( R^{n+1} \) is contained in some \( \mathcal{C}_e \) however, so \( \phi = 0 \).

**Nonlinear perturbations.** The case of the equation
\[(4.5) \quad (\Box - m^2)u = L(u)\]
with \( L \) a possibly nonlinear and time-dependent operator can be successfully treated whenever scattering theory exists for (4.5), considered as a perturbation of the free KG equation
\[(\Box - m^2)u = 0.\]
(Strauss \[7\] has discussed the nonlinear scattering problem, and has obtained some sufficient conditions on \( L \) for existence of the wave operators.) Recall that the wave operators \( W_{\pm} \) are constructed as follows: If \( u(t) \) is a normalizable solution of (4.5), consider the function \( u_* \), obtained as the solution to the KG equation with Cauchy data \( u(s), u'(s) \) at time \( t = s \). Then \( W_{\pm}u = \lim_{s \to \pm \infty} u_* \), assuming that this strong limit in the Hilbert space of normalizable KG solutions exists.

**Theorem 4.2.** If \( W_- \) exists and is 1-1, and \( u \) is a normalizable solution of (4.5) vanishing on the solid backward light cone, then \( u = 0 \).
If \( W_+ \) exists and is 1-1, and \( u \) is a normalizable solution of (4.5) vanishing on the solid forward light cone, then \( u = 0 \).

This theorem is an immediate consequence of our earlier results and the following

**Lemma.** If \( u = 0 \) on the backward (forward) light cone, then so does \( W_- u \) (\( W_+ u \)).

**Proof.** Let \( \mathcal{C} \) denote the solid backward light cone, \( \mathcal{C}_e \) the cone translated through time \( s \), and \( D_* = \mathcal{C} \cap - \mathcal{C} \). With \( u_* \) defined as above, the vanishing of \( u \) on \( \mathcal{C} \) and the hyperbolic propagation prop-
The property of the KG equation imply that \( u_s = 0 \) on \( D_s \) for all \( s < 0 \). Hence \( W_-u \) vanishes on \( \mathcal{C} = \bigcup_{s < 0} D_s \).

(The same proof works for \( W_+u \), of course.)

5. Causal algebras of field operators. Consider the quantized free scalar meson field of mass \( m > 0 \). (See [5].) Mathematically, we have a map \( f \mapsto R(f) \) from \( C_c^\infty(R^4) \) to self-adjoint operators on a complex Hilbert space \( K \) satisfying the usual physical desiderata (commutation rules, Lorentz-transformation properties, irreducibility) and certain continuity requirements. Let \( W(f) = \exp[iR(f)] \), and denote by \( Pf \) the projection of \( f \in C_c^\infty(R^4) \) onto the KG Hilbert space, i.e. the restriction of the Fourier transform of \( f \) to the mass hyperboloid \( k^2 = m^2 \). By hypothesis, the set of operators \([W(f)]\) is irreducible on \( K \), and the map \( f \mapsto W(f) \) is continuous with respect to the Lorentz-invariant Hilbert topology on \( Pf \) and the weak operator topology on \( W(f) \).

**Theorem 5.1.** If \( R(\cdot) \) is the quantized field for the KG equation of mass \( m > 0 \), then the operators \( R(g) \), with \( \text{Supp}(g) \) contained in a fixed open time-like cone, generate all bounded operators on the field state space \( K \).

**Proof.** From Corollary 3.1, the set of all such \( g \) is strongly dense in the KG Hilbert space \( K_m \). Hence by the continuity of the map \( f \mapsto W(f) \), it follows that the ring of operators generated by \( W(g) \), \( g \) ranging over a dense subset of \( K_m \), is \( B(K) \).

**Bibliography**


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