HOMOGENEOUS ALMOST COMPLEX SPACES OF
POSITIVE CHARACTERISTIC

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In this paper we give a necessary and sufficient condition for a
homogeneous space of positive Euler characteristic to admit an in-
variant almost complex structure (i.a.c.s.), Theorem 4.1. A classifica-
tion of these spaces has already been accomplished by R. Hermann
[3], but no characterization of them has been given, to date. This
work is a portion of the author's dissertation directed by H. C. Wang.

1. Let $G$ be a compact, connected Lie group, and $L$ a closed sub-
group of $G$, $\mathfrak{g}$ and $\mathfrak{l}$ their Lie algebras. Then as is well known, $L$ is
reductive in $G$ and we may write $\mathfrak{g} = \mathfrak{g} + \mathfrak{g}_{1}$, where $\text{Ad} L: \mathfrak{g} \to \mathfrak{g}$.
Moreover by homogeneity, a necessary and sufficient condition that
there exist an almost complex structure on $G/L$, is that there exist a
linear transformation $J: \mathfrak{g} \to \mathfrak{g}$, such that $J^{2} = -I$ ($I$, the identity),
and such that $\text{Ad} lJ = J \text{Ad} l$ for all $l \in L$. See for example, A. Frö-
licher [2].

2. We give first a sufficient condition in the general case.

Theorem 2.1. Let $Z$ be a closed, abelian subgroup of a compact Lie
group $G$, and $L$ the identity component of the centralizer of $Z$ in $G$. If
each 1-dimensional subspace of $\mathfrak{g}$ invariant by $\text{Ad} Z$ is pointwise fixed,
then $G/L$ has an i.a.c.s.

Proof. $Z$ being compact and abelian, $\text{Ad} Z$ is completely reducible;
with notation as in 1, there cannot exist any 1-dimensional subspaces
of $\mathfrak{g}$ reducing $\text{Ad} Z$, for if $X \neq 0$ belonged to such a subspace, then by
hypothesis we would have $\text{Ad} z(X) = X$ for all $z \in Z$, and thus $X \in \mathfrak{z}$,
which is impossible.

If $\mathfrak{g}^{\mathbb{C}}$ denotes the complexification of $\mathfrak{g}$, and $\text{Ad} Z$ and $\text{Ad} L$ are
extended by linearity to $\mathfrak{g}^{\mathbb{C}}$, then we know by Schur's Lemma that
$\text{Ad} Z$ acting on $\mathfrak{g}^{\mathbb{C}}$ has a weight vector $v_{i}$. That is, $\text{Ad} z(v_{i}) = \psi(z)v_{i}$
for all $z \in Z$, $\psi$ being a character on $Z$. Moreover, since $\text{Ad} Z$ is real,
$\text{Ad} z(v_{i}) = \overline{\psi(z)}v_{i}$, thus the weight vectors and roots appear in conju-
gate pairs.

Let $v_{1}, \ldots, v_{n}, \bar{v}_{1}, \ldots, \bar{v}_{n}$ be the weight vectors with characters
$\psi_{1}, \ldots, \psi_{n}, \bar{\psi}_{1}, \ldots, \bar{\psi}_{n}$, where some of the $\psi_{i}$'s may be identical. If

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we let \( U \) be the subspace of \( C^\mathfrak{m} \) spanned by the \( v_i \)'s, then each \( x \in C^\mathfrak{m} \) has a unique expression of the form \( x = v + \bar{v} \), where \( v \in U \) and \( \bar{v} \in \overline{U} \). For putting \( x_j = v_j + \bar{v}_j \), and \( x'_j = iv_j - i\bar{v}_j \) the \( x_j \)'s and \( x'_j \)'s span the irreducible subspaces of \( \mathfrak{m} \). And thus any \( x \in C^\mathfrak{m} \) has a unique expression

\[
x = \sum a_jx_j + \sum b_jx'_j,
\]

so

\[
x = \sum \lambda_jv_j + \sum \bar{\lambda}_j\bar{v}_j.
\]

Furthermore, \( U \) is invariant by \( \text{Ad} \ L \), for if \( z \in Z \), \( l \in L \), then \( \text{Ad} \ z \text{ Ad} \ l(v) = \text{Ad} \ l \text{ Ad} \ z(v) = \psi_z(l) \text{ Ad} \ l(v) \), that is, \( \text{Ad} \ l(v) \) belongs to the character \( \psi_z \), and by hypothesis there can be no real characters, thus \( \psi_z \neq \overline{\psi_z} \). Now defining \( J: \mathfrak{m} \to \mathfrak{m} \) by \( J(x) = J(v + \bar{v}) = iv - i\bar{v} \), defines an i.a.c.s. on \( G/L \).

**Corollary 2.1.** If the number of connected components in \( \text{Ad} \ Z \) is odd, then \( G/L \) has an i.a.c.s., \( G, L, Z \) as above.

**Proof.** Suppose there exists a 1-dimensional subspace of \( \mathfrak{m} \), with basis \( X \), reducing \( \text{Ad} \ Z \), then for each \( z \in Z \), \( \text{Ad} \ z(X) = \psi(z)X \), where \( \psi(z) \) is a homomorphism of \( Z \to \mathbb{Z}_2 = \{ +1, -1 \} \). By continuity \( \psi \) is constant on components of \( \text{Ad} \ z \), and takes the value \( +1 \) on the identity component \( (\text{Ad} \ z)^0 \). It thus induces a homomorphism \( \psi: \text{Ad} \ z/(\text{Ad} z)^0 \to \mathbb{Z}_2 \), but \( \text{Ad} z/(\text{Ad} z)^0 \) is a finite group of odd order, therefore its image is identically 1, so that \( \psi = 1 \), and thus \( \text{Ad} z \) is pointwise fixed on each 1-dimensional subspace reducing it.

3. Next we consider the case that \( L \) is semi-simple.

**Lemma 3.1.** Let \( L, H \) be two closed subgroups of \( G \), \( L \subset H \subset G \). And suppose that \( G/L \) has an i.a.c.s. defined by \( J: \mathfrak{g} \to \mathfrak{m} \), where \( \mathfrak{g} = \mathfrak{g} + \mathfrak{m} \). If further \( \mathfrak{m} = \mathfrak{g} + \mathfrak{m}_1 \), \( \mathfrak{m}_1 \subset \mathfrak{m} \), and \( J(\mathfrak{m}_1) = \mathfrak{m}_1 \), then \( H/L \) has an i.a.c.s.

**Proof.** Since \( H \) is a subgroup of \( G \), \( \text{Ad} L: \mathfrak{g}_1 \to \mathfrak{m}_1 \). Thus applying \( 1 \), we see that \( H/L \) has an i.a.c.s.

**Lemma 3.2.** Let \( L \) be a closed semi-simple subgroup of \( G \), of maximal rank, such that \( G/L \) has an i.a. c.s.; then \( L \) is not a symmetric subgroup.

**Proof.** Let \( \mathfrak{g} = \mathfrak{g} + \mathfrak{m} \), and \( J: \mathfrak{m} \to \mathfrak{m} \) define the i.a.c.s. as in \( \S 1 \). Since the torsion tensor of \( J \) is given by

\[
\mathcal{F}(X, Y) = [JX, JY] - J[JX, Y] - J[X, JY] + [X, Y] \mod \mathfrak{g}
\]

for \( X, Y \in \mathfrak{m} \), if we assume \( G/L \) is symmetric \( \mathcal{F}(X, Y) = 0 \), for
symmetric implies \([\mathfrak{m}, \mathfrak{m}] \subseteq L\). Thus \(J\) defines a complex structure on \(G_0/L_0\) (where subscripts denote identity components), but then a classical result of E. Cartan says that \(L_0\) (and thus \(L\)) cannot be semi-simple.

**Lemma 3.3.** If \(G/L\) has an i.a.c.s. and \(L\) is connected semi-simple of maximal rank, \(Z\) the center of \(L\); then \(\text{Ad}(Z)\) acting on \(\mathfrak{m}\) has no 1-dimensional invariant subspaces.

**Proof.** Suppose by contradiction that \(\mathfrak{r} \subseteq \mathfrak{m}\) is 1-dimensional and reduces \(\text{Ad}(Z)\). Let \(x \in \mathfrak{r}, x \neq 0\), then

\[
\text{Ad} z(x) = \psi(z)x, \quad z \in Z, \text{ and } \psi \text{ is a real character on } Z.
\]

Since \(Z\) is compact, either \(\psi(Z) = 1\), or \(\psi(Z) = \{+1, -1\}\). From Borel-deSiebenthal [1], we know that \(L\) is the connected component of the centralizer of its own center \(Z\), and thus \(\mathcal{L} = \{u : u \in \mathfrak{g}, \text{Ad } Zu = u\}\). If \(\psi(Z) = 1\), then \(\mathfrak{r} \subseteq \mathcal{L}\), which is impossible. Thus \(\psi(Z) = \{1, -1\}\).

Let \(\mathcal{Q} = \{y : y \in \mathfrak{m}, \text{Ad } z(y) = \psi(z)y, z \in Z\}\) be the weight space corresponding to the character \(\psi\). Then \(J(\mathcal{Q}) = \mathcal{Q}\), for if \(y \in \mathcal{Q}\),

\[
\text{Ad} z J(y) = J \cdot \text{Ad} z(y) = J \psi(z)y = \psi(z)Jy.
\]

Furthermore, \(\text{Ad} L(\mathcal{Q}) = \mathcal{Q}\), for if \(y \in \mathcal{Q}\), \(l \in L\), then

\[
\text{Ad} z \text{Ad } l(y) = (\text{Ad } l) \psi(z)y = \psi(z) \text{Ad } ly.
\]

But then

\[
[\mathcal{L}, \mathcal{Q}] \subseteq \mathcal{Q}.
\]

Finally if \(y, y' \in \mathcal{Q}\), \(\text{Ad} z[y, y'] = \psi(z)^2[y, y'] = [y, y']\), that is, \([y, y']\) is fixed under all \(\text{Ad } z\), so that

\[
[y, y'] \in \mathcal{L}, \text{ that is } [\mathcal{Q}, \mathcal{Q}] \subseteq \mathcal{L}.
\]

Therefore, \(\mathfrak{h} = \mathcal{L} + \mathfrak{q}\) is a subalgebra and \(H/L\) is a symmetric space, \(H = \exp \mathfrak{h}\). Since \(H\) is of maximal rank, \(H\) is closed by the theorem in §8 of [1]. By Lemma 3.1, \(H/L\) has an invariant almost complex structure, and \(H/L\) is nontrivial since \(\mathfrak{q} \neq 0\), for \(\mathfrak{r} \subset \mathfrak{q}\). This contradicts Lemma 3.2, thus Lemma 3.3 is proven.

**Lemma 3.4.** Under the hypothesis of Lemma 3.3, \(\text{Ad}(Z)\) cannot be a group of order \(2^k, k > 0\).

**Proof.** (a) If \(k = 1\), \(G/L\) is symmetric and this is impossible by Lemma 3.2.

(b) Suppose the lemma is true for all \(k \leq n\), and consider the case that \(\text{Ad } Z\) is of order \(2^{n+1}\). Since \(\text{Ad } Z\) contains a subgroup of order 2,
there exists an element \( z_0 \in Z \) such that \( \text{Ad} \, z_0 \neq 1 \) and \( \text{Ad} \, z_0^2 = 1 \). Let \( G^* \) be the 1-component of the centralizer of \( z_0 \) in \( G \). Then \( L \subseteq G^* \), and we have

\[
G = \mathfrak{L} + \mathfrak{M}, \quad \mathfrak{M} = \mathfrak{q}_1 + \mathfrak{q}_2, \quad G^* = \mathfrak{L} + \mathfrak{q}_1,
\]

where \( \text{Ad} \, z_0 \big| \mathfrak{q}_1 = 1 \) and \( \text{Ad} \, z_0 \big| \mathfrak{q}_2 = -1 \), \( G^* \) being the Lie algebra of \( G^* \). Moreover \( J(\mathfrak{q}_1) = \mathfrak{q}_1 \) for if \( m \in \mathfrak{q}_1 \), \( (\text{Ad} \, z) J(m) = J(\text{Ad} \, z)m = J(m) \).

Similarly we show that \( \text{Ad} \, L(\mathfrak{q}_1) = \mathfrak{q}_1 \).

\( G^*/L \) has i.a.c.s. by Lemma 3.1. But since \( L \subseteq G^* \), \( Z \) is the center of \( L \) in \( G^* \). Since \( \text{Ad} \, z_0 \big| G^* = 1 \), \( \text{Ad} \, Z \big| G^*_L \) is of order \( 2^m, m \leq n \). Thus by the inductive hypothesis \( m = 0 \) and \( G^* = L \). But then \( G/L \) is a symmetric space, contradicting Lemma 3.2. Thus \( G/L \) cannot have invariant almost complex structure, and the lemma is proved.

**Theorem 3.5.** Let \( G \) be a compact connected Lie group and \( L \) a connected semi-simple subgroup of \( G \) of maximal rank; then \( G/L \) has invariant almost complex structure if and only if \( L \) is the identity component of the centralizer in \( G \) of a finite subgroup \( F \) of its center, such that \( \text{Ad}(F) \) is of odd order.

**Proof.** The sufficiency of this condition was proved in Theorem 2.1. To prove the necessity, let \( Z \) be the center of \( L \); then by the fundamental theorem of Abelian groups we may write \( \text{Ad} \, Z = \text{Ad} \, F_1 \times \text{Ad} \, F_2 \), where \( \text{Ad} \, F_1 \) is of odd order and \( \text{Ad} \, F_2 \) is of order \( 2^k \).

Let \( G_1 \) be the identity component of the centralizer of \( F_1 \) in \( G \). We shall show that \( G_1 = L \). Suppose on the contrary that \( L \subseteq G \). Then \( G_1/L \) has i.a.c.s. by Lemma 3.1. If \( \text{Ad}(F_3) : \mathfrak{g}_1 \to \mathfrak{g}_1 \) is identically 1, then we are done because \( G_1 \) centralizes both \( F_1 \) and \( F_2 \) and by the result of Borel and de Siebenthal [1], \( G_1 = L \). But \( \text{Ad}(F_1) \) is identically 1 on \( \mathfrak{g}_1 \), and \( \text{Ad}(F_2) \), if nontrivial on \( \mathfrak{g}_1 \), must be of order \( 2^m \), for some \( m > 1 \). But then \( \text{Ad} \, Z \) acting on \( \mathfrak{g}_1 \) would be of order \( 2^m \), which is impossible. Therefore \( \text{Ad}(F_2) \equiv 1 \) on \( \mathfrak{g}_1 \).

**4. Theorem 4.1.** Let \( G \) be a compact connected Lie group, \( L \) a connected subgroup of maximal rank; then \( G/L \) possesses an invariant almost complex structure if and only if \( L \) is the connected component of the centralizer in \( G \) of a finite subgroup \( F \), of its center, such that \( \text{Ad}(F) \) is a finite group of odd order.

The sufficiency of this condition follows from Corollary 2.1. In this general case when \( L \) is not necessarily semi-simple, we proceed by proving the following two lemmas.

**Lemma 4.2.** Let \( G \) be compact, connected, \( L \) a connected subgroup of maximal rank such that \( G/L \) has an i.a.c.s.; if \( L = T^* \times L_1 \), local direct
product, where \( T^a \) is toral and \( L_1 \) is connected semi-simple, then \( L \) is the centralizer in \( G \) of \( T^a \cup F_1 \), where \( F_1 \) is a finite subgroup of \( L \), such that \( \text{Ad}(F_1) \) is of odd order.

PROOF. \( L = T^a \times L_1, \mathcal{L} = R^a \oplus \mathcal{L}_1 \), let \( G_2 = \text{centralizer of } T^a \text{ in } G \), then \( G_2 \) is a connected subgroup of maximal rank in \( G \), and

\[
G_2 = T^a \times G_3, \quad \mathcal{G}_2 = R^a \oplus \mathcal{G}_3,
\]
where \( G_3 \) is closed in \( G_2 \).

By hypothesis there exists \( J \), commuting with \( \text{Ad}(L) \) such that

\[
J: \mathcal{G}_2/\mathcal{L}_1 \to \mathcal{G}_3/\mathcal{L}_1,
\]
but \( \mathcal{L}_1 \subset \mathcal{L} \), and there exists an isomorphism

\[
\phi: \mathcal{G}_2/\mathcal{L} \to \mathcal{G}_3/\mathcal{L}_1.
\]

Define \( J' \) on \( \mathcal{G}_3/\mathcal{L}_1 \) by \( J' = \phi J \phi^{-1} \), then \( J' \) defines an almost complex structure on \( G_3/L_1 \). Then, by 3.5, \( L_1 \) is the connected component of the centralizer in \( G_3 \) of a finite subgroup \( F_1 \) of its center, such that \( \text{Ad}(F_1) \) is of odd order.

Now, if \( x \in L_1 \), then \( x = t_{i_1} \), where \( t \in T^a \) and \( l_{i_1} \in L_1 \). If \( f_{i_1} \in F_1 \), then

\[
(t \cdot l_{i_1})f_{i_1} = tf_{i_1} = f_1(tl_{i_1}) \text{ since } t \text{ is central, and } f_{i_1}l_{i_1} = f_1f_{i_1} \text{ since } l_{i_1} \in L_1. \]

Thus \( x \) commutes with \( F_1 \), and \( x \) commutes with \( T^a \).

Conversely, if \( x \) belongs to the connected component of the centralizer of \( T^a \cup F_1 \), then \( x \in G_2 \), so that \( x = tg_{i_1} \). Moreover, since \( x \) commutes with \( F_1 \), we have \( x f_{i_1} = f_1 x \), so that \( t g_{i_1} f_{i_1} = f_1 t g_{i_1} f_{i_1} \), so that \( g_{i_1} f_{i_1} = f_1 g_{i_1} \), thus \( g_{i_1} \in L_1 \), and \( x \in T^a \times L_1 \), that is, \( x \in L_1 \).

Lemma 4.3. Let \( G \) be compact, connected Lie group and \( L \) a subgroup of \( G \), such that \( L \) is the centralizer in \( G \) of a toral subgroup \( T^a \). Then there exists an \( x \in T^a \) such that (1) \( L \) is the connected component of the centralizer of \( x \) in \( G \), and (2) \( \text{Ad}(x) \) is of odd order.

Proof of (1). \( T^a \subset T^b \), where \( T^b \) is a maximal torus of \( G \). If \( \theta_i, i = 1, \cdots, m \), are the root forms, or angular parameters of \( T^a \), and \( R^b \) is the covering space of \( T^a \), then \( R^a \), the covering space of \( T^a \), is characterized by \( R^a = \{ x : \theta_i(x) = 0, i = 1, \cdots, m', m' \leq m \} \), where \( \theta_i, i = 1, \cdots, m' \), are angular parameters of \( T^a \). If \( x \in T^a \), and \( Z(x) \) denotes the connected component of the centralizer of \( x \) in \( G \), then \( L \subset Z(x) \). Moreover \( Z(x) \) is minimal if \( \theta_i(x) \neq 0, i = m' + 1, \cdots, m \). But each plane \( \theta_i(x) = 0, i = m' + 1, \cdots, m \), determines a hyperplane of \( R^a \), and the union of these finitely many hyperplanes cannot be all of \( R^a \), thus there exist \( x \in R^a, \theta_i(x) \neq 0, i = m' + 1, \cdots, m \).

Proof of (2). Given any Euclidean space \( R \), the set of elements with coordinates whose denominators are all odd is dense in \( R \), because the set of points with rational coordinates is dense in \( R \). And
if \( x = (r_1, \ldots, r_n) \) where \( r_i = p_i / q_i, p_i, q_i \) relatively prime, is a point with rational coordinates and
\[
x_m = \left( \frac{2m p_1}{2mq_1 + 1}, \frac{2m p_2}{2mq_2 + 1}, \ldots, \frac{2m p_n}{2mq_n + 1} \right),
\]
then \( \lim_{m \to \infty} x_m = r \).

Now let \( \theta_i, i = 1, \ldots, b \), be a basis for \( \mathbb{R}^b \), consisting of root vectors. Thus if \( x \in \mathbb{R}^b \), then \( \langle \theta_i, x \rangle = \theta_i(x) \) defines an inner product on \( \mathbb{R}^b \), and if \( x = \sum x_i \theta_i \), \( \langle \theta_i, x \rangle = x_i \) so that all \( \theta_i(x) \equiv 0 \) mod 1, if \( x \) has integral coordinates in this basis, that is, \( \text{Ad} x = 1 \).

Now if \( x \in T^* \), such that \( Z(x) = \emptyset \), then it does not lie on any of the planes \( \theta_i(x) = 0, i = m' + 1, \ldots \), which is a closed set, and there exists a point \( z \) with rational coordinates, with odd denominators, arbitrarily close to \( x \), so that \( Z(x) = \emptyset \). But \( \text{Ad}(z^k) = 1 \), where \( k \) is the least common multiple of the denominators of the coordinates of \( z \). And, \( \text{Ad}(z^k) = \text{Ad}(z) \).

Combining the last two lemmas, we have that since \( L \) is the centralizer of \( x \in T^* \), such that \( \text{Ad}(x) \) is of odd order and \( L \) is the connected component of the centralizer of \( F_1 \), then \( L \) is the connected component of the centralizer, in \( G \), of the group generated by \( x \) and \( F_1 \), whose adjoint is of odd order.

**Bibliography**